The Logarithmic Chicken or the Exponential Egg: Which Comes First?

Marshall Ransom, Senior Lecturer, Department of Mathematical Sciences, Georgia Southern University
Dr. Charles Garner, Mathematics Instructor, Department of Mathematics, Rockdale Magnet School
Laurel Holmes, 2017 Graduate of Rockdale Magnet School, Current Student at University of Alabama

Background:

This article arose from conversations between the first two authors. In discussing the functions $\ln(x)$ and e^x in introductory calculus, one of us made good use of the inverse function properties and the other had a desire to introduce the natural logarithm without the classic definition of same as an integral. It is important to introduce mathematical topics using a minimal number of definitions and postulates/axioms when results can be derived from existing definitions and postulates/axioms. These are two of the ideas motivating the article. Thus motivated, the authors compared manners with which to begin discussion of the natural logarithm and exponential functions in a calculus class.

A related issue is the use of an integral to define a function g in terms of an integral such as $g(x)=\int_c^x f(t)\,dt$. We believe that this is something that students should understand and be exposed to prior to more advanced "surprises" such as $\mathrm{Si}(x)=\int_0^x \frac{\sin(t)}{t}\,dt$. In particular, the fact that $\ln(x)=\int_1^x \frac{1}{t}\,dt$ is extremely important. But must that fact be introduced as a definition? Can the natural logarithm function arise in an introductory calculus course without the classic definition in many Calculus texts given by $\int_1^x \frac{1}{t}\,dt$? Can use of the classic definition of a derivative of a function, combined with properties of that function, lead to a different way to approach the applications of calculus to the logarithmic and exponential functions? Do prior precalculus understandings of logarithms and inverse functions help us to approach calculus? And, related to this exposition, does the Mean Value Theorem (MVT) actually have a useful application? Perhaps some answers to these questions are presented below.

Logarithms Before Calculus Existed:

Values of trigonometric functions, to many decimal places, were calculated and available hundreds of years ago. Although these were not necessarily in a form we would recognize in the modern era, precision in calculations was becoming more and more important during the 16^{th} and 17^{th} centuries. This was due in no small part to observations made by astronomers. Using this data in ways that required further calculations was tedious, laborious and time-consuming. Enter John Napier and Henry Briggs. Napier's work involved the use of a base and an exponent, and then using the exponents to aid in calculations. The basis for this early work with what we know today as logarithms was closely related to $1/e$ in one case and was 10 the other case. Today, we would not recognize the approaches taken by these earlier mathematicians. More on the history of logarithms is presented in the next section, due to the work of Dr. Garner's student, Laurel Holmes.

A Brief History of Logarithms:

Logarithms are presently defined as being the power to which a particular base must be raised in order to produce a given number, but why is this important? How, exactly, did logarithms even become a part of our present-day mathematics?

The history of logarithms, and the history of mathematics in general, is not one that is often considered by students of the subject. For most, the detailed account of how the logarithm came to be, just as any other topic in mathematics, is one that is seemingly unimportant in being able to learn the concept. As unimportant as it may seem, the origins of logarithms are entirely essential to understanding them as a mathematical topic.

The first publication about logarithms was released in 1614 by a Scottish mathematician named John Napier. Interestingly enough, Napier began his life as a religious activist who fully expected his name to be remembered for his Protestant efforts against the Catholic Church. However, during a revolutionary time in the scientific community, he found himself drawn to new mathematical endeavors.

With new discoveries involving planetary motion and mechanics as well as events such as the circumnavigation of the globe, the seventeenth-century scientific community was accruing a great deal of numerical data. Napier took up the challenge of finding a way to manage the burdensome amount. His result? The logarithm.

While it is unknown exactly how John Napier began his work that would ultimately lead to the revolution that was the logarithm, it is believed that his initial ideas may have stemmed from the addition and subtraction rules for the sines of angles and Michael Stifel's work in geometric and arithmetic progression. Knowledge of geometry, trigonometry, and kinematics also helped Napier in developing logarithms.

It is important to note Napier's kinematic approach to making his tables of logarithms. His consideration of velocity and distance allowed him to ultimately conclude that a point on a line moves geometrically as long as its velocity remains proportional to its distance from the right end of the line. This conclusion was necessary for his process for inventing the logarithm as well as for forming his definition of the logarithm. Napier's goal was to make the sines of angles easier to calculate. To that end, Napier defined the logarithm of any sine to be the number which increased arithmetically with a constant velocity as the radius decreased geometrically during the time in which the radius decreased to a given number.

Napier invented his logarithms through a process of creating two number lines – one increasing in an arithmetic sequence and the other decreasing with a geometric sequence from the right end of the line. He took much of his inspiration from the Powers Tables, and his greatest struggle with initially developing the logarithm was choosing a base that was small enough, but not too small, to fill in the gaps of these tables.

Due to his concerns for minimizing the use of the decimal fraction in his work – a concept which had only recently been introduced into Europe – Napier selected $1–10^{-7}$ as the base for his tables, and then multiplied this by 10^7. Because the predominant goal of Napier was to lessen the work involved in trigonometric calculations, he found his base by dividing the radius of the unit circle into ten-million parts, a common practice in trigonometry at the time. By subtracting the radius of the unit circle and the unit found by dividing the radius, Napier calculated the number closest to 1, and he used this number as the common ratio in the construction of his logarithm tables.

Napier spent the following twenty years of his life subtracting each successive term's 10^{7th} part from the initial 10^7 – that is, by computing $10^7(1–10^{-7})^L$ for L = 1, 2, 3, …, 100 – yielding his first table with one hundred one terms. Next, he created a table with fifty-one terms. This table also started with 10^7, but Napier found the proportion for it by calculating the ratio of the previous number and the first one in his initial table. Napier created three more tables with similar ratios. The last table, which had sixty-eight terms, had a final entry that was nearly half of Napier's original number. Napier called his invention the logarithm, meaning ratio number. Related to Napier's choice of a base is the fact that $\left(1-10^{-7}\right)^{10^7} \approx \frac{1}{e}$, an important connection to what we know today as the natural logarithm. However, someone else had another idea that became a useful manner in which logarithms could be applied to calculations for over 300 years.

After reading Napier's publication "Description of the Marvelous Canon of Logarithms", an Englishman by the name of Henry Briggs recognized the great benefit that the logarithm could have on scientific calculations of the time. The logarithm tables – which were used by finding the logarithm of a desired number and using it to perform calculations to obtain antilogarithms – allowed scientists to multiply any numbers together by simply performing addition. Briggs began working to improve the original ideas of the logarithm, and in the early 1600s, he visited Napier in Scotland to collaborate on logarithms.

It was during the collaborations that significant changes were made to Napier's logarithm, and two forms of the logarithm were created. Briggs convinced Napier to modify his logarithm, with base $1/e$, by using a base ten system meaning log10 = 1. Together, Napier and Briggs adjusted the natural logarithm by comparing points L and x on a graduated straight line. They designated point L for the logarithm in uniform motion from negative to positive infinity. Point x was used as a representation of sine in motion from zero to infinity with its speed proportional to its distance. The common logarithm was produced by limiting the motion of the points using L = 1 at $x = 10$ with the condition that L = 0 when $x = 1$.

The change of the base differentiated Napier's original logarithm into what is now known as the natural logarithm or Napierian logarithm. Because the base ten system came at the suggestion of Briggs, logarithms now using base 10 are referred to as Briggsian logarithms or common logarithms.

While Napier ingeniously invented the concept of the logarithm, it was the writings of Briggs that expounded on the new concept and significantly contributed to the widespread use of logarithms in Europe during the seventeenth century. Briggs' tables of common logarithms, which were completed in 1628 by Adrian Vlacq, also became the new basis for logarithm tables that were used well into the twentieth century.

<u>Precalculus Understandings:</u>

Logarithms were well known prior to the time of Newton and Leibnitz. The basic properties of logarithms should be known to students PRE-calculus, or at the very least, be simple to review in a calculus class. These include $\ln(ab) = \ln(a) + \ln(b)$, $\ln\left(a^p\right) = p\ln(a)$ and $\log_b(a) = \frac{\log(a)}{\log(b)}$. Any use of calculus involving a natural log should imply these properties.

Mean Value Theorem Application:

Too often, the MVT is seen by students only as showing us a magical place where the slope of a tangent line is the same as the slope of a secant line. We use the MVT as follows: if $f'(x)=0$ on (a, b) and $x_1 < x_2 \in (a,b)$, then for any $c \in (x_1, x_2)$, $0 = f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \rightarrow f(x_2) = f(x_1) = C$, a constant. We define the function $h = f - g$. If $f'(x) = g'(x)$ and $h' = 0$ then $h = C \rightarrow f = g + C$. We use this fact, that if functions have equivalent derivatives the functions differ by a constant, several times below.

The functions e^x and ln(x) in Calculus:

The purpose of this section of the article is to establish the derivatives of e^x and ln(x) as well as to derive the integral form of ln(x) and show that this integral satisfies the known precalculus properties of the logarithm. Part I shows that the base e is needed in order for the exponential function to have a derivative equal to itself. Part II introduces the inverse of the function e^x and its derivative. Part III shows the equivalence of the inverse of e^x to the well-known integral. Part IV shows that ln(e) = 1. Part V calculates the derivative of $y = a^x$, and Part VI verifies that the precalculus properties of the logarithm follow from its integral form. The corollary to the MVT mentioned above is used once in Part III and several times in Part VI.

Part I: Calculus with $f(x) = a^x$, $a > 0$ and $a \neq 1$; we establish a special value of a:

1. $\dfrac{d}{dx}a^x = \lim\limits_{h\to 0}\dfrac{a^{x+h} - a^x}{h} = a^x \lim\limits_{h\to 0}\dfrac{a^h - 1}{h}$. The function $y = a^x$ will have itself as its derivative if and only if $\lim\limits_{h\to 0}\dfrac{a^h - 1}{h} = 1$.

2. $\lim\limits_{h\to 0}\dfrac{a^h - 1}{h} = 1 \rightarrow$ for $h \approx 0$, $\dfrac{a^h - 1}{h} \approx 1$. Solving for a we obtain $a \approx (1+h)^{1/h}$.

3. If $\lim\limits_{h\to 0}\dfrac{a^h - 1}{h} = 1$, then $a = \lim\limits_{h\to 0}(1+h)^{1/h}$ which is equivalent to $\lim\limits_{x\to\infty}\left(1 + \dfrac{1}{x}\right)^x$.

4. This famous number is known as e. We have that $\dfrac{d}{dx}e^x = e^x$.

5. The number a which makes $\lim\limits_{h\to 0}\dfrac{a^h - 1}{h} = 1$ is often discussed in precalculus classes, sometimes using experimentation with a calculating device to estimate $\lim\limits_{x\to\infty}\left(1 + \dfrac{1}{x}\right)^x$. This number is $a = e \approx 2.71828$.

Part II: The inverse of e^x

1. Since $e \neq 1$, $y = e^x$ is monotonic and must have an inverse. The range of this function is $y > 0$. Let $L(x)$ be the inverse of e^x. The domain of this function must be $x > 0$.

2. Let $u = f(x)$ be a non-constant, differentiable function of x.
Assuming that $u > 0$ we can write $u = e^{L(u)}$ because e^x and $L(x)$ are inverses.
Taking the derivative of both sides, we have:

3 $\frac{du}{dx} = \frac{d}{dx} e^{L(u)} = e^{L(u)} \frac{d}{du} L(u) \frac{du}{dx} \rightarrow \frac{du}{dx} = e^{L(u)} \frac{d}{du} L(u) \frac{du}{dx}$. Setting equal to 0 and factoring:

4. $\frac{du}{dx}\left(e^{L(u)} \frac{d}{du} L(u) - 1 \right) = 0$. This implies:

(i) $\frac{du}{dx} = 0$ which is only true at critical points of u since u is not constant.

(ii) $e^{L(u)} \frac{d}{du} L(u) = 1 \rightarrow \frac{d}{du} L(u) = \frac{1}{e^{L(u)}} = \frac{1}{u}$, valid for all x in the domain of u because $u > 0$ in that domain.

5. We have established that $\frac{d}{du} L(u) = \frac{1}{u}$.

Part III: $L(x)$ as an integral:

1. Because $L(x)$ and e^x are inverses and $e^0 = 1$, we have that $L(1) = L\left(e^0\right) = 0$.

2. Because $\frac{d}{du} L(u) = \frac{1}{u}$, we must have $\int \frac{1}{u} du = L(u) + C$.

3. Therefore, $\int_1^x \frac{1}{u} du = L(x) - L(1) = L(x) - 0 = L(x)$.

4. We have established that $L(x)$, the inverse of e^x, satisfies $L(x) = \int_1^x \frac{1}{u} du$.

Part IV: Another way to define e:

1. Because $L(x)$ and e^x are inverses, $L(e) = L\left(e^1\right) = 1$.

2. Therefore, e defined as $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ in Part I above has the following property:

$$L(e) = \int_1^e \frac{1}{u}\,du = 1$$

3. $L(x)$ is usually written as ln(x) and is known as the "natural" log with base e.

Part V. What if $y = a^x$ and $a \neq e$?

1. We use a precalculus property of logarithms to calculate the log of both sides of $y = a^x$.
We have $\ln(y) = \ln\left(a^x\right) = x\ln(a)$.

2. We calculate derivatives of both sides of $\ln(y) = x\ln(a)$. We use the chain rule and the fact that $\frac{d}{du}\ln(u) = \frac{d}{du}L(u) = \frac{1}{u}$ from Part III above. This gives us the following result:

3. $\frac{d}{dx}\ln(y) = \frac{d}{dx}x\ln(a) \rightarrow \frac{1}{y}\frac{dy}{dx} = \ln(a) \rightarrow \frac{dy}{dx} = y\ln(a) = a^x\ln(a)$.

4. Therefore, $\frac{d}{dx}a^x = a^x\ln(a)$. For example, $\frac{d}{dx}5^x = 5^x\ln(5)$.

Part VI. The precalculus properties of the logarithm apply to the function $\ln(x) = L(x) = \int_1^x \frac{1}{t}\,dt$, $x > 0$:

1. $\frac{d}{dx}L(x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$ due to the Fundamental Theorem of Calculus.

We use several times below this fact and that $\frac{d}{dx}F(x) = \frac{d}{dx}G(x) \rightarrow F(x) = G(x) + C$.

2. As a basic property of the definite integral, we have that

$$\boxed{\ln(1) = L(1) = \int_1^1 \frac{1}{t}\,dt = 0}$$

3. We calculate the derivative of both sides of $\ln(kx) = L(kx) = \int_1^{kx} \frac{1}{t}\,dt$:

$$\frac{d}{dx}L(kx) = \frac{d}{dx}\int_1^{kx} \frac{1}{t}\,dt = \frac{1}{kx}k = \frac{1}{x} = \frac{d}{dx}L(x)$$

Since $\frac{d}{dx}L(kx) = \frac{d}{dx}L(x)$, we must have $L(kx) = L(x) + C \to L(k) = L(1) + C \to C = L(k)$.

Therefore, $$\boxed{\ln(kx) = L(kx) = \ln(x) + \ln(k)}$$

4. We calculate the derivative of both sides of $\ln(x^p) = L(x^p) = \int_1^{x^p} \frac{1}{t}\,dt$:

$$\frac{d}{dx}L\left(x^p\right) = \frac{d}{dx}\int_1^{x^p} \frac{1}{t}\,dt = \frac{1}{x^p}px^{p-1} = p\frac{1}{x} = \frac{d}{dx}pL(x) \text{ because } \frac{d}{dx}L(x) = \frac{1}{x}$$

Since $\frac{d}{dx}L\left(x^p\right) = \frac{d}{dx}pL(x)$, we must have $L\left(x^p\right) = pL(x) + C \to L(1) = pL(1) + C \to C = 0$.

Therefore, $$\boxed{\ln\left(x^p\right) = L\left(x^p\right) = pL(x) = p\ln(x)}$$

5. We calculate the derivative of both sides of $\ln\left(\frac{x}{k}\right) = L\left(\frac{x}{k}\right) = \int_1^{x/k} \frac{1}{t}\,dt$:

$$\frac{d}{dx}L\left(\frac{x}{k}\right) = \frac{d}{dx}\int_1^{x/k} \frac{1}{t}\,dt = \frac{1}{x/k}\frac{1}{k} = \frac{1}{x} = \frac{d}{dx}L(x)$$

Since $\frac{d}{dx}L\left(\frac{x}{k}\right) = \frac{d}{dx}L(x)$, we must have $L\left(\frac{x}{k}\right) = L(x) + C \to L\left(\frac{1}{k}\right) = L(1) + C \to C = L\left(\frac{1}{k}\right)$.

Using #3 and #4 above, we have $L\left(\frac{x}{k}\right) = L(x) + L\left(\frac{1}{k}\right) = L(x) + L\left(k^{-1}\right) = L(x) - L(k)$.

Therefore, $$\boxed{\ln\left(\frac{x}{k}\right) = L\left(\frac{x}{k}\right) = \ln(x) - \ln(k)}$$

6. We leave it to the reader to establish the change of base property.

Summary:
Exponential and logarithmic functions are fundamental in mathematics. Applying basic calculus to an exponential function of a single, real variable allows us to connect these two functions without defining the natural log as an integral. All properties of logarithms follow from this work. The equality $\ln(x) = \int_1^x \frac{1}{t}\,dt$ is not to be ignored. It is certainly an important way in which functions can be defined; and also, it is important to calculating in calculus. We present a manner in which this can be derived rather than used as a definition.

Bibliography

Boyer, C. B. (2011). *A history of mathematics*. (3r[d] ed.) U.C. Merzbach (Ed.). Hoboken, NJ: Wiley.

Briggs, H. (2001). In *Encyclopedia Britannica online.* Retrieved from https://www.britannica.com/biography/Henry-Briggs

Clark, K. M. & Montelle, C. (n.d.). *Logarithms: The Early History of a Familiar Function - John Napier Introduces Logarithms.* Retrieved from http://www.maa.org/press/periodicals/convergence/logarithms-the-early-history-of-a-familiar-function-john-napier-introduces-logarithms

Katz, V. J. (2009). *A history of mathematics: an introduction.* (3[rd] ed.). Boston, MA: Pearson Education, Inc.

Logarithm. (2017). In *Encyclopedia Britannica online*. Retrieved from https://www.britannica.com/topic/logarithm

Maor, E. (1994). *e: The story of a number.* Princeton, NJ: Princeton University Press.