A New Twist on an Old Box

Abstract

The purpose of this paper is to describe the mathematics that emanate from the construction of an origami box. We first construct a simple origami box from a rectangular sheet and then discuss some of the mathematical questions that arise in the context of geometry and algebra. The activity can be used as a context for illustrating how algebra and geometry, like other branches of mathematics, are interrelated.

A New Twist on an Old Box

The Common Core State Standards put a tremendous amount of emphasis on conceptual understanding (CCSSI, 2010). Origami provides a powerful context for conceptual understanding of mathematical ideas. Among other things, origami gives our students ready-made manipulatives that can be used to visualize abstract mathematical ideas in a concrete manner (Haga, 2006; Hull, 2006). For instance, when one creates a box from a rectangular sheet of paper, the box becomes the object that can be manipulated and analyzed, and abstract concepts like length, width, height, volume, and surface area become something that one can "touch." When students have objects that they have created, students communicate better with one another and with their teacher. Moreover, paper folding in general is essentially mathematics in action. When one is folding paper, she or he is playing with mathematical concepts like perpendicular bisection, angle bisection, and properties of right isosceles triangle, just to name a few of the mathematical concepts that are inextricably connected to paper folding (Tubis & Mills, 2006). In fact, it actually becomes fairly difficult to separate paper folding from mathematics. Due to the link between origami and art, origami can additionally be used to inspire artistic-minded students to think mathematically. Lastly, origami creates a powerful context for the application of Howard Gardner's theory of Multiple Intelligences (Gardner, 2006; Wares, 2013). Gardner's theory of Multiple Intelligences incorporates several other dimensions of intelligences besides *linguistic* and *logical-mathematical* intelligence. Gardner identified the following nine intelligences: *linguistic*, *logical-mathematical*, *bodily-kinesthetic*, *spatial*, *musical*, *interpersonal*, *intrapersonal*, *naturalist*, and *existential intelligence* (Gardner, 2006).

In this paper folding activity, we learn to fold an origami box, and discuss the mathematics embedded in the box. No experience in origami is needed to construct this box. However, it is important to make the creases sharp and accurate. Figure 1 illustrates the two types of creases that are formed when a piece of paper is folded. The constructed box will be a prism with a rectangular base. Figure 2 shows a photograph of the box that we will be making.

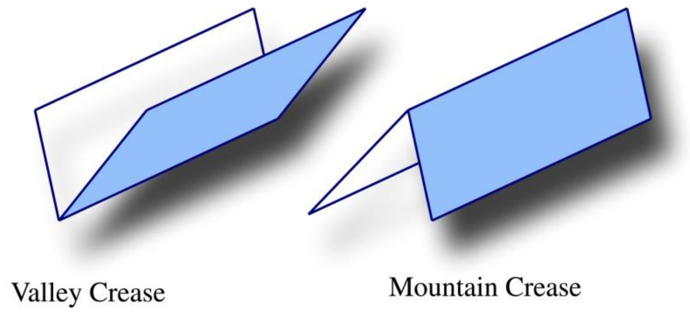

Figure 1. The two types of creases in paper folding.

Figure 2. This is a picture of the box that we will be making.

We will use two sheets of standard 8.5- by 11-inch copy paper to construct the box. Only one sheet will actually become the box itself; we will call that sheet the *origami sheet*. The other sheet will be used as a measuring tool; we will call that sheet the *measuring sheet*. Ideally, the origami sheet should have at least one fancy side; however, even an 8.5- by 11-inch sheet out of the recycling bin will be sufficient for the measuring sheet (the measuring sheet will be discarded during the process, so it does not have to be decorative).

Let us now follow the 18 steps to construct the box. Pictures for Steps 1 through 6 are shown in Figure 3. Pictures for Steps 7 through 13 are shown in Figure 4. Pictures for Steps 14 through 18 are shown in Figure 5. A video showing how to fold this box is also available at the following link: https://youtu.be/1i8zlVkly30

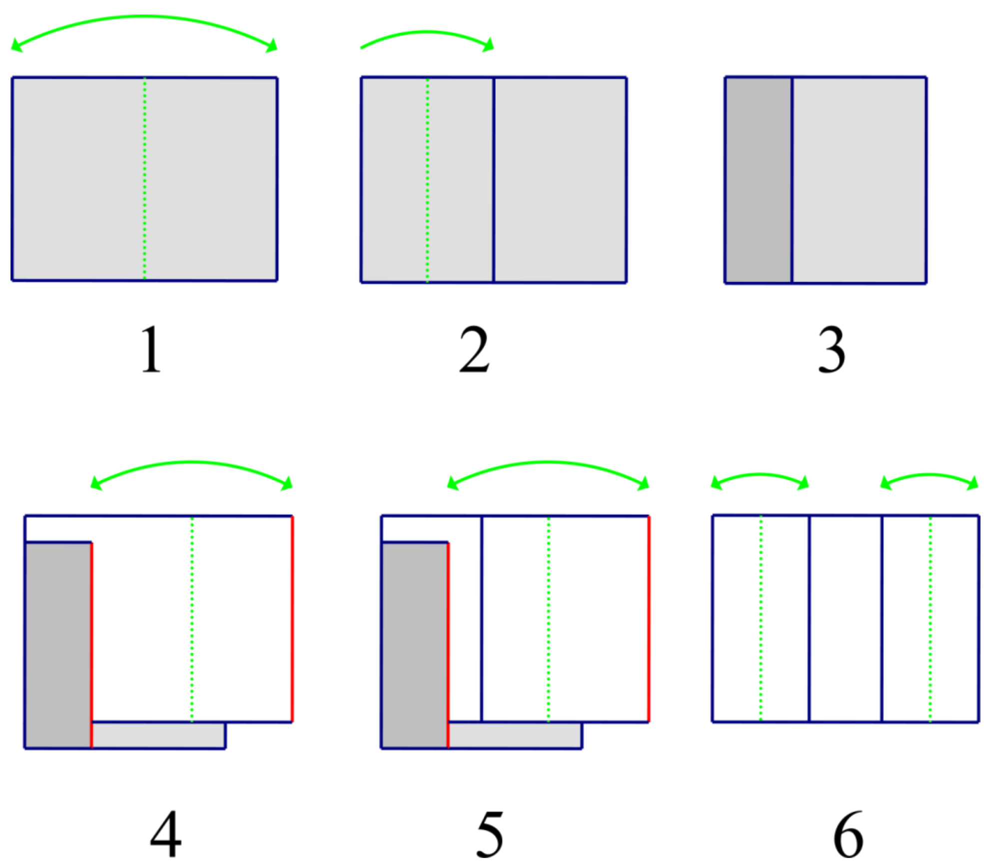

Figure 3: Pictures for Steps 1 through 6.

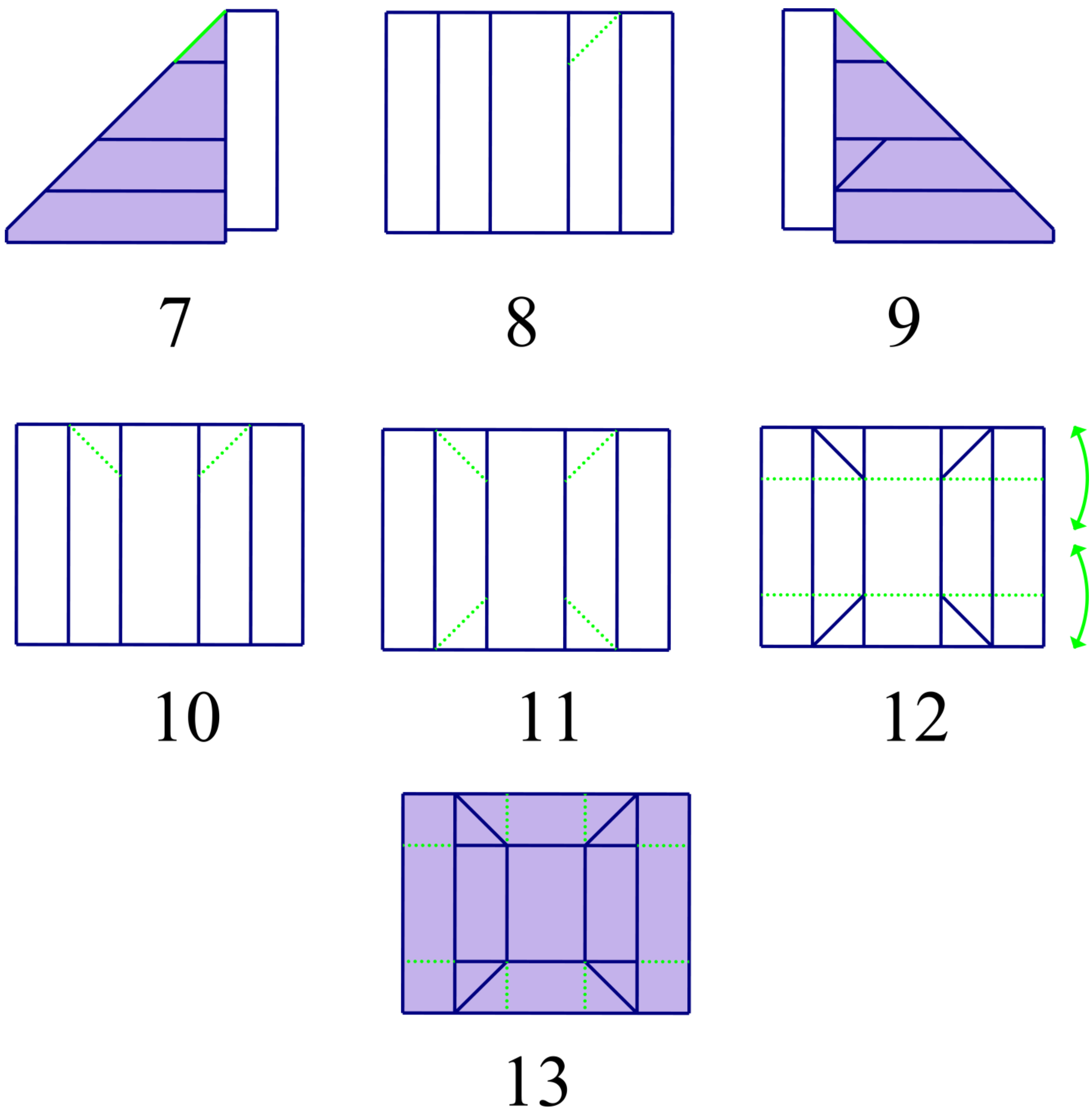

Figure 3: Pictures for Steps 7 through 13.

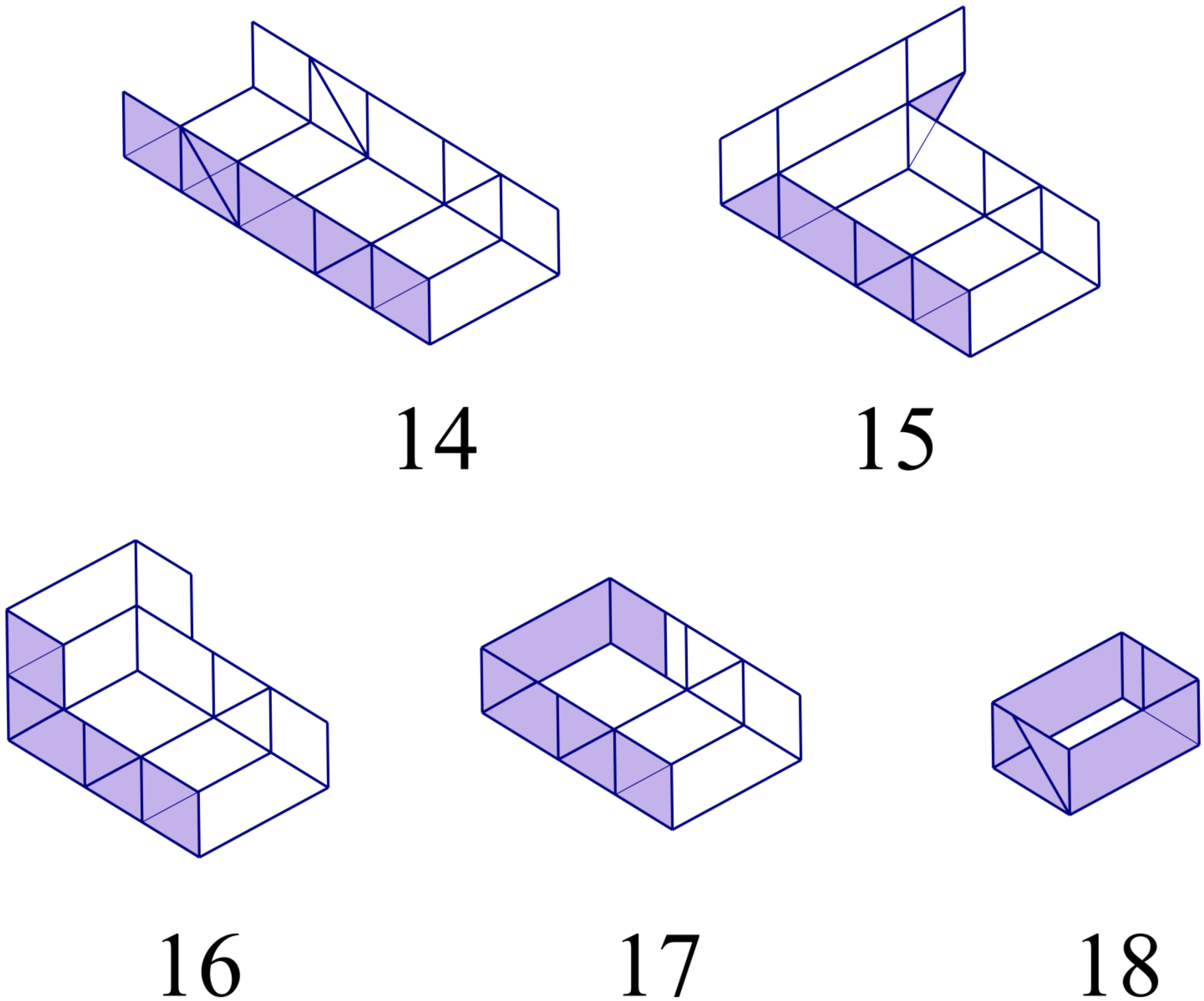

Figure 5: Pictures for Steps 14 through 18.

Step 1: Start with the measuring sheet. By folding and unfolding the two shorter edges of the measuring sheet onto each other, create a valley crease right through the middle of the measuring sheet. This crease must be parallel to the shorter edges of the rectangular measuring sheet.

Step 2: Use one of the shorter edges of the measuring sheet and the crease created in Step 1 to make another crease that splits the measuring sheet into ¼ and ¾ parts. This crease must also be parallel to the shorter edges of the rectangular measuring sheet.

Step 3: Fold ¼ of the measuring sheet over the crease created in Step 2.

Step 4: Take the origami sheet with the plain side up and slide it all the way between the two layers of the measuring sheet. In other words, one of the shorter edges of the origami sheet is pushed against the crease created in Step 2. Fold and unfold the uncovered shorter edge of the origami sheet against the shorter edge of the measuring sheet that is on top of the origami sheet to make another valley crease on the origami sheet.

Step 5: Take the other shorter edge of the origami sheet and push it against the crease created in Step 2. Fold and unfold the uncovered shorter edge of the origami sheet against the shorter edge of the measuring sheet that is on top of the origami sheet to make another valley crease on the origami sheet. Discard the measuring sheet. The measuring sheet will not be needed any longer.

Step 6: Start with the plain side of the origami sheet up. Fold and unfold the left shorter edge onto the crease on the left-hand side of the origami sheet to make one more valley crease. Fold and unfold the right shorter edge onto the crease on the right-hand side of the origami sheet to make one more valley crease.

Step 7: Fold the left-hand side of the top edge of the origami sheet over the crease on the right-hand side to make a short slant valley crease as shown in the picture for Step 8. This slant crease should be between the two creases on the right-hand side as shown in the picture for Step 8.

Step 8: Unfold the origami sheet so that the plain side is up.

Step 9: Fold the right-hand side of the top edge of the origami sheet over the crease on the left-hand side to make a short slant valley crease as shown in the picture for Step 10. This slant crease should be between the two creases on the right-hand side as shown in the picture for Step 10.

Step 10: Unfold the origami sheet so that the plain side is up.

Step 11: Rotate the shape around its center by 180 degrees, and repeat Steps 7 through 10 to make two shorter slant valley creases.

Step 12: Make a horizontal crease by folding and unfolding the top strip of the origami sheet that contains the two slant creases. This horizontal crease must pass through the terminal points of the two top slant creases. Make another horizontal crease by folding and unfolding the bottom strip of the origami sheet that contains the other two slant creases. This horizontal crease must also pass through the terminal points of the two bottom slant creases.

Step 13: Flip the origami sheet over so that the fancy side is up. Make valley creases along the dotted lines. Note that mountain creases already exist along these dotted lines; you just need to change the orientation of these creases from mountain to valley.

Step 14: Start with the plain side of the origami sheet up. Lift the top and bottom parts of the sheet along the creases that are parallel to the longer edges of the origami sheet. These vertical strips will form two of the walls of the box that is being constructed.

Step 15: Fold along two of the short slant creases as shown and lift up the part of the sheet that contains one of the shorter edges of the origami sheet.

Step 16: Fold the parts of the paper that contain the two short slant creases inward as shown.

Step 17: Tuck the excess paper in carefully so that the creases line up neatly inside the partially constructed box. This should complete the third wall of the box.

Step 18: Use steps very similar to Steps 15 through 18 with the other lose end of the partially constructed origami sheet to complete the box.

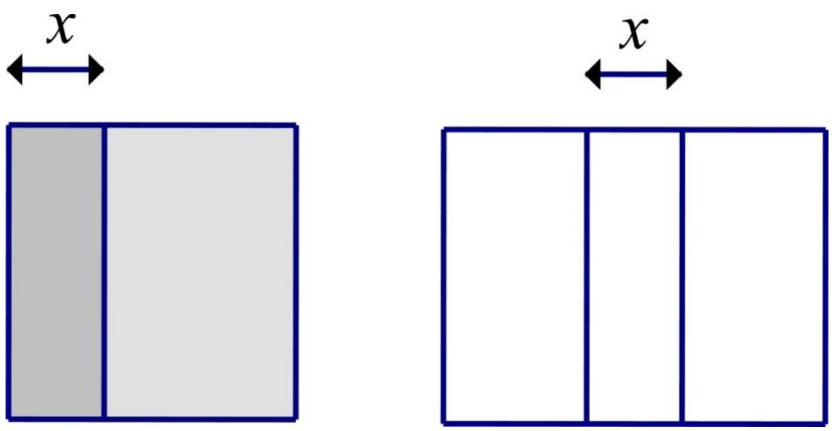

Figure 6. The measurement sheet is shown on the left and the origami sheet is shown on the right.

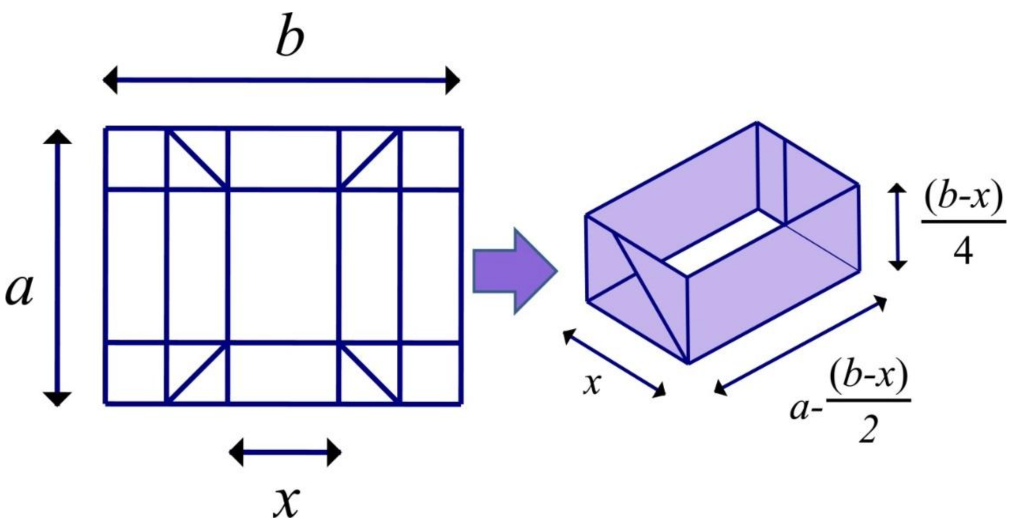

Figure 7. Connections between the crease marks created on the origami sheet and the dimensions of the constructed box.

Now that we have the box, let us carefully analyze the constructed box. Open the box up so that we can analyze the crease marks created by the folds. One of the questions that comes to mind is: How can we determine the dimensions of the constructed box if we know the dimensions of the rectangular sheet? We can show that if the width of the folded portion of the *measuring sheet* is x inches (see picture on the left in Figure 6), then the distance between the first two creases of the *origami sheet* is also going to be x inches (see picture on the right in Figure 6). Suppose our origami paper is an a inches by b inches rectangular sheet. Then the dimensions of the base of the constructed box must be x inches by $a - \frac{(b-x)}{2}$ or $\frac{2a-b+x}{2}$ inches, and the height of the constructed must be $\frac{b-x}{4}$ inches (see Figure 7). The expressions for the dimensions of the box can easily be verified using Euclidean geometry. Students should be encouraged to come up with these expressions on their own. Let $V(x)$ denote the volume of the constructed box.

Therefore,

$$V(x) = x\left(\frac{2a-b+x}{2}\right)\left(\frac{b-x}{4}\right) = \frac{2abx - b^2x - 2ax^2 + 2bx^2 - x^3}{8}.$$

Therefore,

$$V'(x) = \frac{2ab - b^2 - 4ax + 4bx - 3x^2}{8}.$$

$V'(x) = 0,$

$$\Rightarrow \frac{2ab - b^2 - 4ax + 4bx - 3x^2}{8} = 0$$

$$\Rightarrow x = \frac{2b - 2a \mp \sqrt{4a^2 - 2ab + b^2}}{3}$$

Suppose b = 11 and a = 8.5, then $x = \frac{2b - 2a \mp \sqrt{4a^2 - 2ab + b^2}}{3} = \frac{5 \mp \sqrt{223}}{3} = 6.6444$. Note that we are only considering the positive value of x. Therefore, when the distance between the first two parallel creases on the origami sheet is about 6.64 inches, the volume of the constructed box is the largest. Figure 8 shows the graph $V(x)$. Moreover, when a = 8.5m and b = 11, $V(6.6444) = 45.7417$.

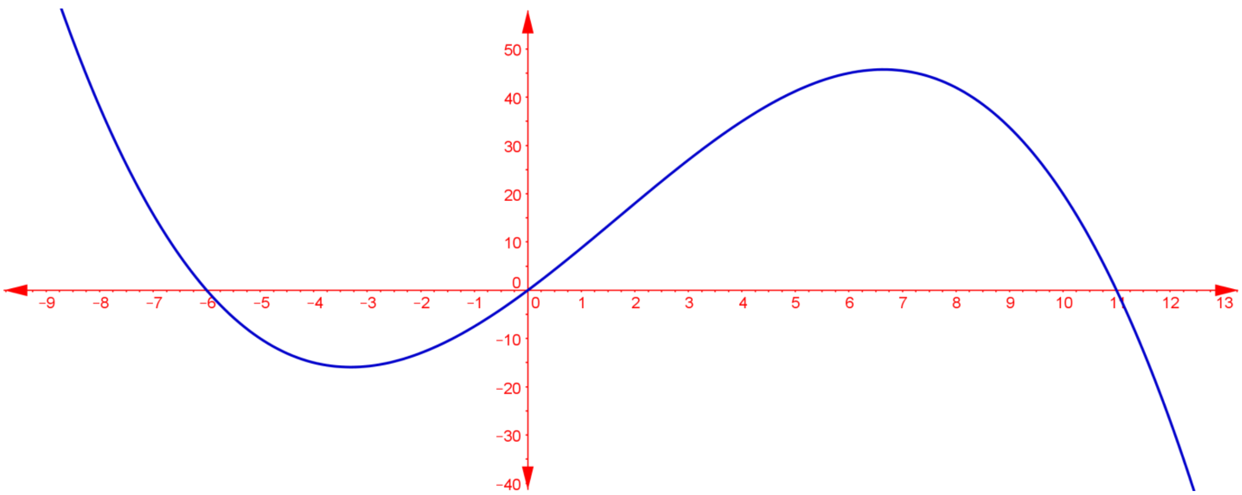

Figure 8. The graph of $V(x)$, with a = 8.5 and b = 11.

One of the strengths of paper folding activities is that they can be easy to understand and model, yet at the same time, the solutions are not always obvious. Since students have a concrete object (the folded paper) in front of them, the level of intellectual engagement with the task is heightened and the quality of communication in the classroom becomes richer during the lesson. By its nature, origami creates a context for rich educational discourse.

Since mathematics is a cultural endeavor, it is ideal to teach mathematics as it manifests in the context of various cultures (D'Ambrosio, 2001). Appreciation of cultural diversity is not only important for minority groups, but it is also important for the dominant ethnic group in any society because even the members of the dominant ethnic group will be working in an environment that is becoming increasingly diverse (Shirley, 2001). The use of origami in mathematics classroom can provide a powerful context for the appreciation of the cultural diversity present in our world.

The Common Core (CCSSI, 2010) describes the following as the *Standards for Mathematical Practice*:

Make sense of problems and persevere in solving them

Reason abstractly and quantitatively

Construct viable arguments and critique the reasoning of others

Model with mathematics

Use appropriate tools strategically

Attend to precision

Look for and make use of structure

Look for and express regularity in repeated reasoning

The author believes all of the above *Standards of Mathematical Practice* can be implemented in a well-orchestrated and well-designed activity involving origami. Activity described in this paper incorporates most, if not all, of the *Standards for Mathematical Practice*. More specifically, the origami activity described in this paper incorporates the following *Standards for Mathematical Content* in high school: *seeing the structure in expressions* (algebra), *interpret functions that arise in applications in terms of the context* (functions), *build a function that models a* *relationship between two quantities* (functions), *explain volume formulas and use them to solve problems* (geometry), *visualize relationships between two-dimensional and there-dimensional objects* (geometry), and *apply geometric concepts in modelling situations* (geometry).

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