Fractions: Not Always a Two-Way Street

Students need a strong understanding of fractions as a launching pad for further progress in mathematics, particularly in algebra, which is a foundation for higher-level mathematics (Wu 2002, Fennell 2007, NMAP 2008, Wu 2014). However, the subject of fractions continues to be difficult for many students (Hecht, Vagi, & Torgeson 2007; Mazzocco & Devlin 2008; NMAP 2008), leading to further implications for their mathematical studies or careers. An effective way to help students understand fractions more deeply is to present a variety of representations, methods, explanations, and justifications (Harvey, 2012; Pantziara & Philippou, 2012, Cramer & Henry, 2002; Siebert & Gaskin, 2006). By using this approach, several different yet equivalent visual representations of a single fraction can be explored in order to help students better understand its value. With this technique in mind, we must examine the opposite direction – that is, for one given visual representation, are there several correct fractions, or is there only *one* correct fraction (or an equivalent one) that represents the visual? In this article, the authors include tasks at the third-grade level and student responses designed to help teachers uncover some issues associated with fraction concepts so that they are better prepared when introducing fractions to their own students.

From Written Fractions to Visual Representations

To assess our students' prior knowledge of fractions, we asked them to draw a representation of $\frac{4}{5}$. **Figure 1** displays some possible student drawings for this task.

Task 1: How can we represent $\frac{4}{5}$?

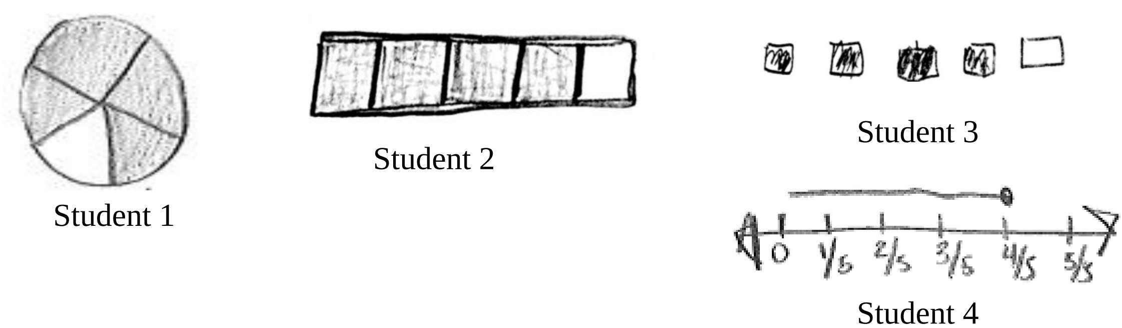

Figure 1. Sample Student Work for Task 1: How can we represent $\frac{4}{5}$?

One such interaction with a student who drew the first picture, a circle area model, is as follows:

Teacher: How did you model $\frac{4}{5}$?

Student 1: I drew a circle and made 5 equal pieces and then shaded 4 of them.

Teacher: Why did you cut the whole into 5 pieces? Why not 4?

Student 1: Because the bottom number tells how many equal pieces the whole is broken into.

Teacher: And the top number?

Student 1: The top number tells how many are shaded.

The dialogue with Student 2 was similar since the student also used an area model to represent $\frac{4}{5}$. The teacher asked the student to clarify why they cut the whole into 5 pieces because of the possible ambiguity in area models. Student 3 used a set model approach and viewed the whole as 5 separate equal pieces and shaded 4 of those pieces. Without any information given about the whole, this drawing can be interpreted as 4 boxes shaded out of 5 total boxes as opposed to thinking of each piece as "one-fifth" in size. When using a set model to represent fractions, some students, unfortunately, focus on the numerator and denominator as simply whole numbers interfering with their ability to see the fraction itself as a number (Siebert & Gaskin, 2006). Student 4's linear model also shows $\frac{4}{5}$ because this student has an understanding that 1 can be represented by $\frac{5}{5}$ and each iteration is $\frac{1}{5}$. On a number line, the whole is the length of the line segment from 0 to 1, so there is less confusion about the identity of the whole when using this model. In other words, when using set and area representations, the whole may not always be explicit to students. For brevity of this paper, the authors will focus on area models.

From Visual Representations to Written Fractions: Introducing Ambiguity

As shown in **Figure 1**, when students were given the fraction $\frac{4}{5}$, they drew different but equivalent models for $\frac{4}{5}$ by interpreting 5 as *the number of equal-sized pieces the whole is being broken into* and 4 as *the number of those equal-sized pieces being used*. Conversely, if given a visual representation, can students name the correct fraction or an equivalent one that represents the picture? To address this question, we intentionally created ambiguity by posing Task 2 (**Fig. 2**).

Task 2: Lily had her birthday party at Statesboro Pizza Palace. Below is the left-over pizza that Lily boxed up at the end of the party. Name a fraction to represent the sausage pizza.

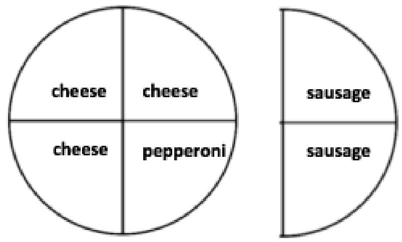

Figure 2. Task 2

As shown in **Table 1**, possible responses from students were 2, $\frac{2}{6}$ (or $\frac{1}{3}$), $\frac{2}{4}$ (or $\frac{1}{2}$), $\frac{2}{8}$ (or $\frac{1}{4}$). Can all these answers be correct? If so, how? If students explore all these responses, will they be more or less certain of their answer? The students' uncertainty of the correct response could be based on the "ambiguous whole" (Danielson, 2016).

Table 1. Possible Responses for Task 2 and Reasoning

Responses	**Reasoning**
2	Each piece is a whole, and there are 2 pieces of *sausage pizza*.
$\frac{2}{6}$ (or $\frac{1}{3}$)	All of the leftovers (6 pieces) is the whole, and the amount of sausage pizza is 2/6 of *the leftovers*.
$\frac{2}{4}$ (or $\frac{1}{2}$)	One pizza is the whole, and the amount of sausage pizza is 1/2 of *a pizza*.
$\frac{2}{8}$ (or $\frac{1}{4}$)	The whole is two pizzas with a total of 8 equal pieces. The amount of sausage pizza is 2/8 of *two pizzas*.

All answers (2, $\frac{2}{6}$, $\frac{1}{3}$, $\frac{2}{4}$, $\frac{1}{2}$, $\frac{2}{8}$, $\frac{1}{4}$) were put on the board and students were asked to determine how other students may have gotten these responses. A discussion follows.

Teacher: Look at all the different responses. Can all these answers be correct?

Student 1: No.

Teacher: Why do you think that?

Student 1: Because these answers are all different, and I know in math there can be only one right answer.

Student 2: But in our group, one person got $\frac{2}{4}$ and another person got $\frac{2}{6}$. We talked about it and agree on both answers.

Teacher: Why did your group agree on both answers?

Student 2: Because I said it was $\frac{2}{4}$ *of a pizza*, and Sally said it was $\frac{2}{6}$ *of the leftovers*.

Teacher: How can that happen?

Student 3: They are looking at different wholes.

Teacher: What do you mean by that?

Student 3: Well, one person is looking at *one pizza* as the whole, and another person is looking at *the leftovers* as the whole.

Teacher: Let’s hear from someone who hasn’t spoken yet. Can these students correctly come up with those answers?

Student 4: Yes, because you didn’t identify your whole in the task.

Teacher: What do you mean?

Student 4: You just asked us to name a fraction to represent the sausage pizza, but you didn’t tell us what we are using as our whole.

Teacher: Using that logic, how could someone get 2 or $\frac{2}{8}$ as a response?

Student 1: They could say $\frac{2}{8}$ of *two pizzas.*

Teacher: What about 2.

Student 5: 2 *pieces.*

Teacher: So what do you notice about this task?

Student 1: We need to be specific about what our "whole" is.

Why is it Important to be Precise?

The Mathematical Standards for Practice (SMP) (CCSS, 2017) include eight practices that help determine how students should be engaged in the content and should be included in lesson plans for effective mathematics instruction. SMP #6 "Attend to precision" (CCSS, 2017) was a strong focus, as the authors unpack what students need to understand to be able to successfully complete this task. Attending to precision is not just getting a correct numerical answer, but also identifying units. Looking back at Task 2, we get multiple answers because the whole is not precisely defined. Students answered 2 *slices* of sausage pizza because they interpreted each piece as a separate "whole." If a student observed that there were 2 equal pieces of sausage pizza out of the 6 left-over pieces, he or she might say $\frac{2}{6}$ (or $\frac{1}{3}$) *of the leftover pizza*. In this case, the student viewed all the leftovers (6 pieces) as the whole. Some students gave $\frac{2}{4}$ (or $\frac{1}{2}$) *of a pizza* for an answer because they noticed that one pizza was cut into 4 equal-sized pieces and there are 2 pieces of sausage pizza. Our last example of $\frac{1}{4}$ or $\frac{2}{8}$ could be explained if students saw 2 pizzas as the "whole." Even though two pizzas were not shown in the picture, some students completed the pizza including the sausage portion to show that two pizzas were broken into 8 pieces or 4 pieces for each pizza. Since there were 2 pieces of sausage pizza, students said $\frac{2}{8}$ (or $\frac{1}{4}$) *of 2 pizzas*.

To Be Precise, Identify the Whole When Naming Fractions

In order to help students evaluate each of the answers from Task 2, they must identify the whole in each case. According to social media, one common misconception regarding the current state of mathematics teaching is that any response a student gives is correct. This, however, is not true because $\frac{1}{3}$ is not equal to $\frac{1}{2}$. Even when including the units ($\frac{1}{3}$ *of the leftover pizza* versus $\frac{1}{2}$ *of a pizza*), the responses are different and need to be addressed as such. The goal of this article is to help teachers understand how students arrive at these different responses so that we can be more precise with our language as we ask students to work through tasks. If the whole is not explicitly stated, as in Task 2, students may interpret the tasks differently. Therefore, if teachers plan to introduce an ambiguous task, such as Task 2, they should expect multiple answers and require their students to justify their answers by including units with their numeric answer. If a teacher specifies the whole within the task, as shown both Tasks 3 and 4 (**Fig. 3 & 4**), there is only one correct (or equivalent) answer.

Unlike Task 1, Tasks 2, 3 and 4 have specific goals. For Task 2, the ambiguity allows students to discover multiple numerical responses, which leads to a rich discussion about the importance of including one’s choice of the whole with an answer. (Danielson, 2016). For both Tasks 3 and 4 (**Fig. 3 & 4**), the whole is fixed so that multiple responses will be addressed as either correct or incorrect. Each task only differs in the bold print part of the question.

Task 3: Lily had her birthday party at Statesboro Pizza Palace. Below is the left-over pizza that Lily boxed up at the end of the party. Name a fraction to represent **how much of the leftover pizza** was sausage.

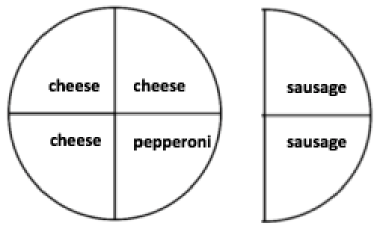

Figure 3. Task 3

In Task 3, some students may respond with $\frac{2}{6}$ of the leftovers, or they may group three pairs of two equal pieces together and interpret 2 sausage pizza pieces as 1 set from 3 sets total to arrive at the answer $\frac{1}{3}$. Other students may break all the current equal pieces in half and write $\frac{4}{12}$. The goal is to allow students flexibility in identifying these fractions and to notice that even though these fractions ($\frac{2}{6}, \frac{1}{3}, \frac{4}{12}$) look different, they are all equivalent since they all refer to the same whole (leftover pizza), unlike the responses in Task 2.

Task 4: Lily had her birthday party at Statesboro Pizza Palace. Below is the left-over pizza that Lily boxed up at the end of the party. Name a fraction to represent **how much of a pizza** was sausage.

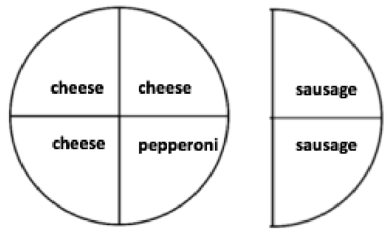

Figure 4. Task 4

How does Task 3 differ from Task 4? The question in Task 4 is *how much of a pizza.* Since the "whole" in these two tasks are different, correct student responses to these tasks were also different. In Task 4, a correct student response was $\frac{2}{4}$ of a pizza (or $\frac{1}{2}$ of a pizza). When the whole changed, the written fraction to represent the picture changed as well.

Let's revisit Task 2. When students first defined the whole when working through this task, they realized that the ambiguity disappeared. Students may ask "Which whole should I choose?" For Task 2, the goal was for students to engage in a rich discussion about the importance of the whole and to recognize they had flexibility with choosing their whole. They could give the answer of $\frac{2}{6}$ *of the leftovers*, but they could **not** give the answer $\frac{2}{6}$ *of a pizza*, or $\frac{2}{6}$ *of two pizzas*. This is important because we asked students to give *complete* answers, which included both the numerical response and the units.

Connecting to the Common Core State Standards

How do these activities relate to the content standards? In grade 3, students should, "Understand a fraction $\frac{1}{b}$ as the quantity formed by 1 part when a whole is partitioned into *b* equal parts; understand a fraction $\frac{a}{b}$ as the quantity formed by *a* parts of size $\frac{1}{b}$" (CCSSI, 2017, p. 24). When working through the tasks, students who drew area models made sense of fractions in Task 1 (**Fig. 1**) by discussing how they broke the whole into a certain number of equal pieces or 5 in this case. Then they shaded 4 of those $\frac{1}{5}$ -sized pieces. Similarly, the number line modeled the whole, 1, as $\frac{5}{5}$. This student drew a line representing 4 of those $\frac{1}{5}$-sized pieces. While Task 1 presented students with a fraction and asked them to draw a representation of that fraction, Tasks 2, 3, and 4 (**Fig. 2-4**) included a representation and asked students to name a written fraction that represented the picture. Being aware of issues that may arise with different tasks helps teachers anticipate potential misconceptions that could occur during the class discussion, which is one of the five important parts for productive classroom discussion (Smith & Stein, 2011). Making sense of the whole in early grades helps students develop an understanding of future concepts. For example, even though fraction equivalence is not explicitly taught in this lesson, students make sense of the answers $\frac{2}{6}$ and $\frac{1}{3}$ in Task 2. The foundation our students build in grade 3 will either help or hinder their understanding of fractions in later grades.

Suggestions for Fraction Instruction

Making sense of fractions is not always a two-way street. In particular, from a given fraction, one can draw different but equivalent models to represent the fraction but the other way around may not be true. The issue arises when students are given a visual representation and asked to represent the visual with a fraction when the whole is not explicitly specified. Therefore, we would like to offer some suggestions for fraction instruction.

- When given a written fraction, have students draw multiple correct models so that they can make sense of the fraction and be exposed to different representations.
- Pose ambiguous tasks similar to Task 2 to promote rich discussions that help students understand the importance of the whole.
- When presenting students with a visual or word problem involving fractions, include a context where the "whole" is explicitly stated; otherwise, expect multiple answers.
- Require students to give complete answers, which include both a numerical response and unit.

On a final note, the authors encourage teachers to try these tasks in their classrooms and to share any strategies, explanations, and justifications their students use when solving the tasks as well as any misinterpretations and discussions with the goal of deepening our students' fraction understanding.

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