Some History of the AP Calculus Curriculum and Exam 1969-2022

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This article traces a history of changes in Advanced Placement Calculus. Some changes involve topics such as differential equations. Some involve significant changes such as allowing the use of a graphics calculator by students for a few of the questions. Some involve changes in the scoring of free response questions (FRQs), requiring more mathematics in the explanations of solutions. And, believe it or not, some involve less mathematics being shown by students.

The intent of this article is to look at some of the significant changes in AP Calculus, often through the lens of the FRQs. This discussion begins in 1969, the first year of two separate courses, AB and BC. For information describing the AP program and calculus prior to 1969, the reader is referred to the three articles below.

"Meeting the Challenge of AP Calculus, III: Foundations" Bressoud, 2010

"The strange role of calculus in the United States" Bressoud, 2020

Eric Rothschild
The *History Teacher*, Feb., 1999, Vol. 32, No. 2, Special Issue: Advanced Placement (Feb., 1999), pp. 175-206
Available at Four Decades of the Advanced Placement Program on JSTOR

I. <u>How many FRQs are on the test?</u>

From 1969 through 1982 there were seven free response questions (FRQs). In 1983 and 84 there were five, and a scientific calculator was allowed. Since 1985 there have been six FRQs on both AP Calculus exams. In 1993 and 1994 scientific calculators were once more allowed. Beginning in 1995 a graphics calculator has been allowed for some questions. Possibly of more significance than the total number of questions on the exam is an attempt at making all exam questions have "entry-level" parts—parts that should be accessible for AP Calculus students. This approach has been successful in minimizing the number of students who provided little or even no response to the last question or two.

II. <u>Topics dead and gone?</u>

Beginning in 1969 with the first BC Exam, expectations for solutions of differential equations were much more robust than at present. Also, up through the 1997 exam, precalculus material was explicitly tested on both the AB and BC exams. Some examples follow.

1. Differential Equations:
1970BC4 required general solutions of both $y''+25y=0$ and $y''-6y'+25y=0$. 1981BC5 part a asked for a general solution of $y''+y'-6y=0$ leading to the request, in part c, for the general solution of $y''+y'-6y=e^x$. 1975BC6 part a asked for a general solution of $y''-(p+1)y'+py=0$. This 1975 question further asked for a particular solution (under some additional given conditions) and then a limit as $p\to 1$, including a justification for that limit evaluation. 1986BC4 part a asked for a general solution to $y'=2y-5\sin(x)$. 1989BC part a asked for a general solution to $\frac{dx}{dt}-10x=60e^{4t}$. These types of equations require solutions involving use of such methods as a multiplying factor, a characteristic equation, and/or use of the method of undetermined coefficients. These methods are typically studied in a first course dedicated to ordinary differential equations in the undergraduate curriculum in college. As of 1991, these methods of solving differential equations were no longer part of the curriculum for AP Calculus. It should be noted that solving such equations is not a bad way to work with students who are ready for additional topics after administration of the AP Calculus exam. But these types of equations are now "dead" as far as the AP Calculus curriculum is concerned. The current Course and Exam Description (CED) aligns fairly well with traditional college courses: AB with Calculus I and BC with both Calculus I and Calculus II.

After 1991, a differential equation requiring a general solution could be solved using separation of variables on both the AB and BC exams. However, these equations have increasingly been included in application questions, sometimes in a context rather than simply "solve for the function $y=f(x)$ in terms of x." The application could be as straightforward as stating the equation of a tangent line at a point (often serving as the initial condition of the equation, leading to a particular solution). The tangent line equation could also be used to approximate a value of the function at a point near the given initial condition. 2006AB5, for example, also requested the sketch of a slope field at eight indicated points. This 2006 equation was $\frac{dy}{dx}=\frac{1+y}{x}$. The slope field was requested in part a. Part b asked for a particular solution given that $y=f(x)$ and $f(-1)=1$. In order to earn the last point on this problem, students had to give the domain of the solution. That added domain question stumped most students. The domain in this situation needs to be of a continuous function that contains the point $(-1,1)$. This is the domain of one branch of the solution $y=2|x|-1$, seen as a "V" graph, with the one discontinuity at $x=0$. This divides the "V" graph into two branches. The branch containing the initial condition $f(-1)=1$ requires that $x<0$.

For many years (after 1991 and most often without the domain question) a differential equation problem on the exam has required solving the equation using only the technique of separation of variables. Some students, well-trained for the differential equation problem, often got into difficulties on such questions as 2014AB6 in which the particular solution was asked for in part c. A number of students did not carefully read the question and launched straight into solving this in their part a work, sometimes completely missing an opportunity for point or two. Part a actually asked for a sketch of the solution through a given slope field, and part b asked for a tangent line equation and its use in finding an approximation at a nearby point. Students, who started by solving the equation, presented work that was more difficult for readers to score and often left out answers to the first two parts. 2015AB4 presented the equation $\frac{dy}{dx}=2x-y$ and asked four questions, none of which required a general solution. Students who felt obligated to first solve the equation had great difficulty with the question, the equation not solvable by separation of variables (without the clever substitution $u = \frac{dy}{dx}$ and some involved subsequent work). However, all four parts of the problem offered opportunities for earning points without ever solving the equation.

2012AB5 presented the equation $\frac{dB}{dt} = \frac{1}{5}(100 - B)$. This problem was in the context that $B(t)$ gave the weight of a bird t days after it was first weighed. Part a asked about the rate of weight gain at two different times, requiring the *use* of the given derivative. Part b showed a possible graph of B, asking why the graph could not be correct, requiring a calculation of and appeal to $\frac{d^2B}{dt^2}$. Finally, in part c, an initial condition was given and the particular solution was requested.

Another comment about the separable differential equation is informative. For years, the solution to the differential equation was scored on the exam beginning with a point for separating the variables followed by one or two points for the needed antiderivatives. Next, one point was awarded for the timely use of a "+*C*" in student work. This was separate from a point for using the initial condition (merely substituting correctly) and a further point for the final solution. Beginning in 2016, the correct use of "+*C*" had to accompany more in the work in order to earn a point. 2016BC4 part c provides an example of this change in scoring procedure, wherein both "+*C*" *and* the use of the initial condition were required in order to earn just one point.

2. Precalculus Topics:
Explicitly testing precalculus knowledge was part of the curriculum through 1997. These skills are important in analyzing the behavior of functions on a calculus exam. But they are now considered prerequisite knowledge, rather than being singled out to be specifically demonstrated on the exam.

Looking back to 1969AB1, we see students presented with these four functions:
$f_1(x) = x,\ f_2(x) = x\cos(x),\ f_3(x) = 3e^{2x},\ f_4(x) = x - |x|$. The first three of four parts to this question asked if $f(-x) = -f(x)$, does the inverse function exist for all x, and whether or not the functions are periodic. The fourth question asked about continuity at $x = 0$. In 1978AB1, students are given that $f(x) = x^3 - x^2 - 4x + 4$ and in part a are asked for the zeros of $f(x)$, certainly no calculus required. Part c asks for values of a and b if a line tangent to $f(x)$ at (a,b) contains the point $(0,-8)$. The slope of the line containing (a,b) and $(0,-8)$ must be set equal to $f'(a)$ (all right, some calculus) leading to a bit more algebraic work (and finding a root of $2a^3 - a^2 - 12 = 0$) than might be on an exam after 1997. 1980AB3 presents the two functions $f(x) = \ln\left(x^2\right),\ x > 0$ and $g(x) = e^{2x},\ x \geq 0$. The two functions $H(x) = f\left(g(x)\right)$ and $K(x) = g\left(f(x)\right)$ are defined. The first two parts of this question ask for domains of H and K and expressions for H and K without exponential functions. The third part asks for an expression for $f^{-1}(x)$ and its domain. 1993AB5 showed a graph of $f'(x)$, two line segments having endpoints (0, 0) and (2, 0) and intersecting at (1, 1). The domain of $f(x)$ is specified as (0, 2). In part a of this problem, students are asked to write an expression in terms of x for $f'(x)$. Without applying this precalculus skill, it would be almost impossible for students to

answer the subsequent two questions in this problem. These examples show some explicit testing of precalculus topics.

But it should be remembered for a number of reasons that these topics are important prerequisite skills. Here is an example. 2018AB6 gives students the equation $\frac{dy}{dx}=\frac{1}{3}x(y-2)^2$ and in part c asks for a particular solution given that $y=f(x)$ and $f(1)=0$. Correct work separating variables, integrating, and using a constant leads a student to the equation $\frac{-1}{y-2}=\frac{1}{6}x^2+C$. To finish earning the fourth point, the student must substitute the initial condition. But no point is awarded for finding the constant. That effort is included in the solving for y for one more point, and that work does not require calculus. This type of skill (another example is simply solving for x if $f'(x)$ is set equal to 0) is important to review and emphasize when teaching the AP Calculus courses. Precalculus topics are now "dead" as far as being explicitly tested, but are alive and well in many other ways. (For example, being able to deal with compositions and inverses of functions is still very important.)

III. <u>A new emphasis included? What does "accumulation" mean?</u>

Defining a function in terms of the integral of another has often been seen on the exam. For example, see 1995AB6. Since 1995 more applications of a definite integral have been used. The 1995 example provides a graph of the derivative of a function and defines a new function in terms of a definite integral. Questions asked about this "new" function require use of the fundamental theorem but not yet areas seen on the graph. Use of such areas seen in a graph as well as applications of accumulations of "amounts" (whether it be distances, water moving in or out, or accumulations of such things as snow falling) became a staple, an expected question on the exam.

1. Accumulations of area:

The fact is that the definite integral, in its simplest form, is a limit of a Riemann sum. In the case of a continuous function $f(x)$, the Riemann sum calculates the sum of signed areas of rectangles above and below the *x*-axis of a graph in two dimensions. If the function is continuous, the limit of this sum as the width of the rectangles approaches zero is the value of the definite integral of $f(x)$. As one moves from left to right on the *x*-axis, the areas in the Riemann sum "accumulate."

An early example of this is found in 1997 AB/BC5 part a. Students were given a graph of a continuous function f consisting of a semicircle and two line segments on the interval [–2, 5]. The function *g* was defined as $g(x)=\int_0^x f(t)\,dt$. Part a asked for the value of $g(3)$ which required combining ("accumulating") signed areas.

Values of a function defined as *g* is in 1997 are sometimes requested to the left of the initial point as in 2012AB3. Some problems show a graph for which areas cannot be calculated geometrically as in 2013AB3 and 2015AB5. In these cases, areas of regions bounded by the curve and the *x*-axis are given. Other questions can be asked about properties of the function defined as an integral of the function shown in the graph. These questions ask about regions of increase vs. decrease, concave up vs. concave down, max and min, and points of inflection. In such questions, accumulation of areas is not necessarily relevant to the analysis.

1998 AB/BC5 is a wonderful problem for either teaching or student practice. Students are given a function $F(t) = 80 - 10\cos\left(\frac{\pi t}{12}\right)$ that models temperature outside a house over a 24 hour period. In part a students were asked to graph this function over the 24 hour period on a grid provided, and a graphing calculator was allowed. Of interest to the current discussion, in part c students were told that the cost of cooling the house accumulates at \$0.05 per hour whenever the temperature is above 78 degrees. In order to calculate the cost of cooling over the 24 hour period, students had to calculate the integral $\int_a^b .05\left(F(t) - 78\right)dt$ where *a* and *b* are the times when $F(t) = 78$.

If the function is $v(t)$, a velocity function for a particle moving along the *x*-axis, the signed areas represent distances traveled to the right and to the left. The resulting sum in the limit is the definite integral of $v(t)$ and is the displacement of whatever particle is in motion from its initial position over the time interval in question. A definite integral of $|v(t)|$ is, as the limit of the Riemann sum, a representation of all positive distances travelled. Thus $\int_a^b |v(t)|\,dt$ gives the total distance travelled by the particle on the interval $a \le t \le b$. Another example: if rain is falling at a rate given by the function $R(t)$ inches per hour, where *t* is in hours, the integral $\int_0^2 R(t)\,dt$ gives the total amount of rain that has fallen between $t = 0$ and $t = 2$ hours (absolute value not being necessary because there are no negative amounts of rain falling). The value of this integral provides the number of inches of rain fallen as is seen in $R(t)\,dt = ((inches / hour) \times hours) = inches$. These inches correspond to the area under the curve $R(t)$ between $t = 0$ and $t = 2$ hours.

2. Input-output models:

Questions on the exam that can be referred to as input-output models rely on the rates given and the definite integrals thereof (although not strictly upon the idea of accumulation of areas). For example, if the rate of accumulation of rainfall $R(t)$ as above is combined in an exam question with a rate $E(t)$ of evaporation of the rain, questions can be asked about when the total amount of water is a max or min or whether the rate of accumulation at a specific time is increasing or decreasing. Of course, other quantities can provide context for a problem on the exam. For example, 2017AB2 gives a rate $f(t) = 10 + (0.8t)\sin\left(\frac{t^3}{100}\right)$ for $0 < t \le 12$ at which bananas are being removed from a table and another rate $g(t) = 3 + 2.4\ln\left(t^2 + 2t\right)$ for $3 < t \le 12$ at which bananas are added to the table, where the rates are measured in pounds per hour and *t* is in hours. The questions appear to be straightforward, but it turned out that the most vexing part was explaining the meaning of $f'(7)$ in the context of the problem. This is the rate of a rate of change, often difficult to explain in words. Other parts of this question did involve definite integrals as bananas "accumulated." The 2010AB/BC1 problem gives the "output" function as a piecewise defined function describing the rate at which snow was shoveled from a driveway, providing a comparison to the rate at which snow is falling (accumulating) which was given as $f(t) = 7te^{\cos(t)}$ cubic feet per hour.

Summary: Applications in context involving input-output models are now present on the AP Calculus Exam each year. Problems involving an analysis of the properties of functions are also present. The latter often involve reasoning from a given graph of a derivative of a function. Both of these types of problems rely on the idea of "accumulation" as calculated by a definite integral. A more specific use of this word is seen clearly beginning with the 2019 CED, Intro to Unit 6 (Developing Understanding): "Integration determines accumulation of change over an interval, just as differentiation determines instantaneous rate of change at a point."

IV. <u>A rigorous upgrade</u>

Both the AB and BC Calculus courses now (particularly as of 2017) require students to use definitions and theorems to build arguments and justify conclusions. Questions on the exam can ask students to "show" something or "explain why…." or "justify your answer." Expectations of students who are asked to "justify your answer" have increased, certainly since 2005 and much more so in 2017.

1. The irrelevant sign chart:

Prior to 2005 a question may have asked a student to identify a value of x for which a function $f(x)$ has a relative max or min and to justify the answer. Suppose there was a max at $x = 2$. The calculus explanation would be that $f'(x)$ changes sign at $x = 2$ from positive to negative. But a sign chart (it had to be labelled as $f'(x)$) showing the value of 2 on the number line and signs + and – to the left and right of 2 respectively was accepted as justification. Not any more. A significant change took place in 2005, and an article was posted explaining this change and what was expected of students. Students henceforth had to explain in words what information the sign chart provided in order to justify the conclusion. Correct labelling of a sign chart does not necessarily indicate that a student knows what this means in terms of properties of the function under analysis. In fact, student work prior to 2005 sometimes included statements and occasionally those statements and conclusions were incorrect, even in the presence of a correctly labelled sign chart. This article, specific to sign charts, portended an expectation of more explanatory information from student work on the exam. The article can be accessed on AP Central at signcharts2005apcentr_36691.pdf (collegeboard.org).

2. Explain and/or justify:

Suppose the Mean Value Theorem has to be applied to answer a question such as "is there a c on (1, 4) such that $f'(c) = 4$?" Some work showing a difference quotient with correct values was expected, but for justification, if readers merely saw "MVT," that was acceptable. Not any more. Starting with the 2017 exam, references to the hypotheses of such theorems as the MVT have been required. Prior to 2017, a situation that may have involved invoking the Intermediate Value Theorem merely required the presence of "IVT" somewhere in student work. A statement that the function in question was continuous on the appropriate interval would also have sufficed in the past, but it is now a requirement that continuity be mentioned in student work applying the IVT, not just mentioning "IVT." In other words, more words are now required in order to justify answers. It should be noted that naming the theorem is not required, but if named, it must be correct.

3. How to state that hypotheses are satisfied:
Regarding the Intermediate and Mean Value Theorems, certain conditions must be satisfied on an interval. On the AP Calculus Exam, it is usually not necessary to specify these intervals. But if specified, the intervals must be correct. For both the IVT and MVT, continuity is required on a closed interval. For the MVT, it must also be stated that differentiability applies on an open interval. The MVT situation can be even more interesting. If a function is correctly identified as differentiable on a closed interval, then that interval contains the open interval and is a sufficient statement regarding differentiability. However, although differentiability does imply continuity, when applying the MVT students are expected to state that both of these are properties of the function under analysis. In other words, a reader scoring student work can not assume that the statement "… is differentiable on [1, 4]…" means that the student knows that continuity is implied. Explicit statements regarding both continuity and differentiability are required in order for students to earn all available points on the exam when applying the MVT (even when continuity of the function under examination is specified in the stem of the problem).

4. L'Hospital's Rule:
On the AP Calculus Exam FRQs, the need to apply L'Hospital's Rule has arisen in computing $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$. If we have $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)=0$, then L'Hospital's Rule can be applied. This has been the nature of the functions used on the exam FRQs. The limits equaling 0 need to be shown, as of 2017. Following that, the computation of $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ needs to be shown. Even if it is clear that both $f'(a)$ and $g'(a)\neq 0$ exist, it is required since 2017 in invoking this theorem to show work with limit notation.

In the case where the limits of both functions in the ratio $\dfrac{f(x)}{g(x)}$ are 0, the expression $\dfrac{0}{0}$ is often seen somewhere in student work. This in itself is neither necessary nor sufficient in order to show that the limits are both 0. Further, $\dfrac{0}{0}$ is indeed an "expression," sometimes referred to as a "form" and is not a value or a number. Thus, student work showing something like $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{0}{0}$ or $\dfrac{f(x)}{g(x)}=\dfrac{0}{0}$ removes eligiblity to earn all available points for a question requiring L'Hospital's Rule, even if correct limit notation is otherwise used in the presented work.

5. More about continuity beginning in 2017:
As mentioned above, continuity on a closed interval needs to be stated in student work when using either the MVT or the IVT. Again, this continuity can not be inferred by a reader of student work simply because

differentiability has been stated. Despite differentiability being correctly stated in student work, it also needs to be stated by the student that this implies continuity. Sometimes "implication" does not need to be specifically cited; merely the additional fact that continuity is a property of the function does need to be cited. But as requirements for more rigor seem to increase on the exam, it is wise for a student to include such statements as "…g is differentiable and therefore continuous…" or "… g' exists implies g continuous…"

An exception to explicitly stating continuity (and more) about a function is the use of the Extreme Value Theorem (EVT), which states that a function continuous on a closed interval will attain absolute maximum and minimum values somewhere on that closed interval. Thus far on the exam it has not been necessary for a student to state that the function under analysis is continuous. Further, the search for locations where the function attains absolute extrema requires use of a theorem that states that if a derivative exists at a point which is a location of a relative max or min, then the value of the derivative is 0. Currently, students do not need to state that theorem in applying the EVT. The work required for justification is calculating (or sometimes merely stating and using) values of the function at endpoints of the interval and at points where the derivative does not exist or is 0. This is sometimes referred to as "the candidates test." Going into the 2022 exam, it is not expected that more rigor will be required of students when using the EVT. This is somewhat contrary to the phrase in the general instructions for the exam, indicating that justifications "…require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied." Despite the EVT as an exception, this phrase quite well describes the increased expectations for rigor in student work shown on the exam, especially beginning in 2017.

V. 2022…and into the future?

Mention has been made above of known expectations and curriculum guidelines leading to 2022. For example, the CED is a detailed document divided into units and including suggested activities. On the APCentral website is AP Classroom, an interactive source of more than merely assistance to both teachers and students. Help available for the AP Calculus courses is robust. This should be taken advantage of by students and teachers alike. No longer are resources limited to past released exams and a course outline booklet, fondly referred to in years past as "The Acorn Book."

The 2005 article mentioned above regarding sign charts is just one of many helpful articles on various topics. Expect the wealth of material assistance regarding AP Calculus to continue expanding, as it also is for the many other AP courses. Two basic links to the AP Central Calculus Course Pages are:

AP Calculus AB - AP Central | College Board

AP Calculus BC - AP Central | College Board