The Georgia State Mathematics Tournament Volume One, 1977-1983

The Georgia Council of Teachers of Mathematics

The Georgia State Mathematics Tournament, Volume One: 1977-1983

First edition, 2018

Dr. Chuck Garner, Editor
Rockdale Magnet School for Science and Technology
Conyers, Georgia USA

This book was produced directly from the editor's LaTeX files.
Figures were drawn using the TeXdraw package.
Set in *Lucida Bright*, Lucida Sans, and *Lucida Casual*,
designed by Charles Bigelow and Kris Holmes.

Cover image:
"Hyperbolic tiling of (p, q, r) fundamental domain triangles, in Poincaré Disc
projection, colored red for odd reflections" by Tom Ruen
Image from Wikimedia Commons, `commons.wikimedia.org`

Thanks to Angelique Allen of Greater Atlanta Christian School
for providing copies of the 1982 and 1983
State Tournament materials.

All proceeds from the sale of this book go directly to GCTM to support the State Mathematics Tournament.

Dedicated to ***Gladys M. Thomason*** *who first brought the idea of GCTM sponsoring a math tournament to the GCTM Executive Committee in September 1972; to* ***Dr. Bill Bompart*** *who worked from 1974 to 1977, in his capacities as Vice-President and President of GCTM, to establish the tournament; to* ***Dr. Hiram Johnston*** *who, in 1975-1977, chaired the committee that studied the feasibility of a tournament organized by GCTM, and how it should be run; and to* ***Brenda Tapp****, the chair of the first state math tournament committee.*

An Important Message

The records GCTM keeps of the State Tournaments are incomplete! This book only contains the 1982 and 1983 State Tournaments. The other five tournaments from the years 1977, 1978, 1979, 1980, and 1981 are missing. If you have a copy or know where a copy could be found, please contact Chuck Garner at `cgarner@gctm.org`*.*

Preface

This book contains all the problems from Georgia's State Mathematics Tournament from 1982 and 1983. As the older tournaments are found, they will be included in this book. The book is divided into three parts: THE PROBLEMS, THE SOLUTIONS, and THE ANSWERS. The problems are exactly those that appeared at the tournament, although some editing of language was done. In the solutions we present one possible solution to every problem, and sometimes more than one possible solution. The answers are in the back of the book to enable students to check their work without reading the solution. This should encourage students to figure out for themselves how to solve the problem before reading the solution.

Overview. The top thirty-six schools in Georgia are invited to participate in the state math tournament. Always held on the last Saturday in April, Invitations are issued based on results from previous tournaments around the state during the school year. Schools are invited to select a four-person team to compete in a 90-minute, 50-question written test, a 10-question individual ciphering round, and a 12-question team round in which the team of four works the problems together. The team round is a modern addition to the State Tournament. Only the test and ciphering were offered during the years covered in this book.

Logistics. The State Tournament is the responsibility of the Georgia Council of Teachers of Mathematics (GCTM). In particular, it is the pervue of the Vice-President for Competitions and the Tournament Secretary to ensure the tournament takes place. The Tournament Secretary issues the invitations and handles registration. The VP for Competitions is responsible for securing the facilities and trophies for the event as well as establishing the State Tournament Committee, comprised of at least six high school or university mathematics teachers from around the state. The Committee's repsonsibility is to write the test and ciphering problems and to help run things on the day of the tournament.

Format and Scoring. During the years covered in this book, the written test was simply 50 multiple-choice questions.[1] The scoring at the time was that students receive four points for each correct answer, and they lose a point for incorrect responses. Blanks earned zero points. Each student begins with 50 points, so the maximum points available on the test is 250. To avoid ties, certain questions were designated as "tie-breakers" before the tournament, and which questions were designated as such was unknown to the participants. Each correct individual ciphering answer is worth ten points if submitted within the first minute, and five points if submitted in the second minute.

Calculator Usage. No calculators were allowed on the tests or the ciphering during the years included in this book. Today, students may still use any "non-qwerty" and "non-CAS" calculator they wish on the test.

Content. The written test during the years 1977 to 1983 can be likened to a very challenging precalculus test, with other assorted topics thrown in. Indeed, a glance at the index—which is categorized by mathematical topic—reveals many trigonometry, geometry, and advanced algebra problems. (See the index on page 55.) It is interesting to note that the written tests included in this book do *not* conform to the current topic breakdown, which includes more topics from discrete mathematics: number theory, combinatorics, and counting. Even though the content of the tournaments in this book is different from the typical 21st century State Math Tournament, there are still challenging and thought-provoking problems to be enjoyed in this collection.

Winners. Table 1 lists the schools and Table 2 lists the individuals that were crowned state champions from 1977 to 1983.

Future Plans. This book represents the culmination of a five-year plan to publish all extant State Tournaments. We will continue with future books as State Tournaments occur. The next book (Volume Seven) will include the years 2016 to 2020.

[1]The change to the current format of 45 multiple-choice questions and 5 free-response questions occurred in 1994.

1977	Walton High School, Marietta
1978	Henderson High School, Chamblee
1979	Sprayberry High School, Marietta
1980	Wheeler High School, Marietta
1981	Wheeler High School, Marietta
1982	Walton High School, Marietta
1983	Walton High School, Marietta

Table 1: School Champions

1977	Robert Thompson, Carver
1978	unknown[2]
1979	Steve Jones, Sprayberry High School
1980	David DeMille, Wills High School
1981	unknown[3]
1982	Peter Murphy, Wheeler High School
1983	Jim Maloney, Walton High School

Table 2: Individual Champions

Thanks. This book represents the work of many people on the State Tournament Committee over the years. Without the work of these dedicated persons, the tournaments would not have happened: Diane Brewer, Mary Brown, Margie Burge, Robert Catanzano, Rachel Crowe, Don Dorminey, Fay Early, Linda Head, Dr. Jim Hutcheson, Pam Johnson, Dr. Hiram Johnston, Delores Jones, Rita Long, Dr. Dwight Love, Phyllis Praytor, Mike Rogers, Sharon Shadden, Mildred Sharkey, Earl Swank, and Brenda Tapp.[4]

Chuck Garner, Editor
CONYERS GA
JANUARY 2018

[2] Any record of the winner is sadly lost.

[3] Any record of the winner is sadly lost.

[4] This is only a partial list. The files in the possession of the VP for Competitions are tragically incomplete. Only the names of the tournament committee members from 1977, 1978, 1982, and 1983 are preserved. If anyone has any information about the other years, please let the VP for Competitions know!

Contents

Part I

The Problems

1

The 1982 State Tournament

The Written Test

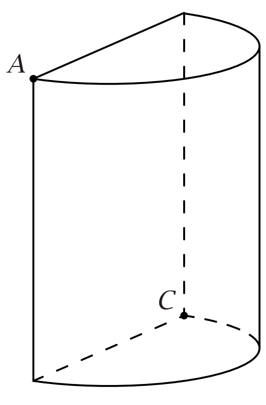
1. The inequality $|x - 4| \leq 5$ can be expressed without absolute values by

(A) $x - 4 \leq 5$ **(B)** $x - 4 \leq -5$ **(C)** $x \leq 9$ **(D)** $-1 \leq x \leq 9$ **(E)** $-9 \leq x \leq 1$

2. Successive discounts of 5%, 10%, and 20% are equivalent to a single discount of

(A) 35% **(B)** 31.6% **(C)** $12\frac{1}{3}$% **(D)** 17.5% **(E)** $11\frac{2}{3}$%

3. What is the shortest distance from A to C along the curved surface? The figure to the right is half a circular cylinder of length 4 and diameter 4.



(A) $4\sqrt{2}$
(B) $2(\pi + 1)$
(C) $2\sqrt{\pi + 1}$
(D) $2\sqrt{\pi^2 + 4}$
(E) 16π

4. John can type a given mailing list in 15 hours and Mary can do the same job in 10 hours. If they did the job working together, it would require

(A) 6 hours **(B)** 12 hours, 30 minutes **(C)** 4 hours, 28 minutes **(D)** 5 hours **(E)** None of these

5. The circumference of a circle is x feet; its area in square yards is

(A) $\frac{x^2}{\pi}$ **(B)** $\frac{x^2}{4\pi}$ **(C)** $\frac{x^2}{12\pi}$ **(D)** $\frac{x^2}{36\pi}$ **(E)** None of these

6. Stan shot a bullet at a target x feet away. The bullet traveled to the target at 2000 feet per second. The sound of the bullet striking the target returned to Stan at 1100 feet per second. Stan heard the bullet strike the target 6.2 seconds after he fired. How far, in feet, was Stan from the target?

(A) 4000 **(B)** 4400 **(C)** 4500 **(D)** 4600 **(E)** 4700

7. Solve for t: $3^{3t} = 9^t$.

(A) $t = 0$ **(B)** $t = 1$ **(C)** No solution **(D)** $t = \frac{3}{2}$ **(E)** None of these

8. Which of the following lines are asymptotes of the graph of the function below?

$$y = \frac{x}{x+4}$$

(A) $x = -4$ only **(B)** $y = -4$ only **(C)** $x = -4$ and $y = -4$ **(D)** $x = -4$ and $y = 1$ **(E)** $x = 0$ only

9. $(x+y)^{-1}(x^{-1}+y^{-1}) =$

(A) $x^{-2} + 2x^{-1}y^{-1} + y^{-1}$ **(B)** $\frac{1}{(x+y)^2}$ **(C)** $\frac{1}{x^2} + \frac{1}{y^2}$ **(D)** $\frac{1}{xy}$ **(E)** None of these

10. Find the area of a triangle whose vertices are $(0,1)$, $(1,4)$, and $(3,6)$.

(A) 7 **(B)** 4 **(C)** -4 **(D)** 2 **(E)** -2

11. The line $\dfrac{x}{3} + \dfrac{y}{4} = 1$ has slope

(A) $\dfrac{4}{3}$ **(B)** $-\dfrac{4}{3}$ **(C)** $\dfrac{3}{4}$ **(D)** $-\dfrac{3}{4}$ **(E)** $-\dfrac{1}{3}$

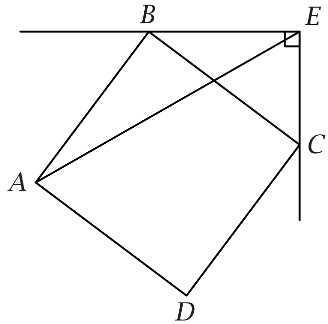
12. Simplfy $\sqrt[n]{3^{n+2} \div 9}$ where $n \geq 2$ is an integer.

(A) $3^{(n+2)/n} \div 9^{1/n}$ **(B)** $\left(\dfrac{1}{3}\right)^{(n+2)/n}$ **(C)** 1 **(D)** $\left(\dfrac{1}{9}\right)^{1/n}$ **(E)** 3

13. Find the limit (if it exists): $\lim\limits_{h \to 0} \dfrac{\sqrt{4+h}-2}{h}$.

(A) 0 **(B)** $\dfrac{1}{2}$ **(C)** $\dfrac{1}{4}$ **(D)** 1 **(E)** Does not exist

14. In the figure, $ABCD$ is a square, $CE = 3$, and $BE = 4$. Find AE.



(A) $\sqrt{65}$ **(B)** $\sqrt{55}$ **(C)** $\sqrt{11}$ **(D)** 9 **(E)** 10

15. The triangle whose vertices are the points $(x, 0)$, $(-x, 0)$, and $(0, y)$ is equilateral if

(A) $x = y$ **(B)** $y = \dfrac{x}{2}$ **(C)** $y = x\sqrt{3}$

(D) $y = \dfrac{x\sqrt{3}}{2}$ **(E)** None of these

16. What is the value printed by the program below?

```
10 LET A = 1
20 LET B = 1
30 FOR J = 1 TO 7
40 LET B = B + A
50 LET A = A + B
60 NEXT J
70 PRINT A
80 END
```

(A) 377 **(B)** 2 **(C)** 1 **(D)** 610 **(E)** 987

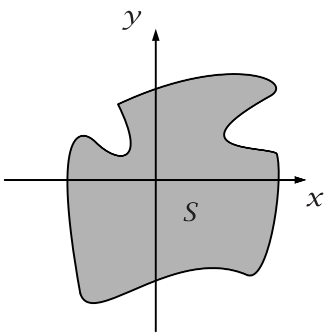
17. Four students enter a classroom and decide to sit in the front row which contains seven seats. With each student sitting in one seat, how many different ways could they be seated?

(A) $\binom{7}{4}$ **(B)** $P(7,4)$ **(C)** 7! **(D)** 4! **(E)** None of these

18. Two dimensions of a rectangular solid are 4 and 6. If the diagonal of the solid is 14, find its volume.

(A) 144 **(B)** 168 **(C)** 288 **(D)** 336 **(E)** 576

19. In the figure shown, shaded region S has an area of 5. What is the area of the region T which consists of all points $(x-5, y+10)$ where (x, y) is in S?



(A) 5 **(B)** 10 **(C)** $10\sqrt{5}$ **(D)** 20 **(E)** $2\frac{1}{2}$

20. If $f(x) = |x - 1|$ and $g(x) = 1 - x^2$, then which of the following is $3f(-2) + 4g(-3)$?

(A) -23 **(B)** -5 **(C)** 13 **(D)** 32 **(E)** -13

21. Solve $3(2x + 7) - 5 = 2(5x - 4) + 4x$.

(A) -1 **(B)** 3 **(C)** No solution **(D)** $\frac{6}{5}$ **(E)** 0

22. What is the sixty-third term of $2, \frac{9}{4}, \frac{5}{2}, \ldots$?

(A) $15\frac{3}{4}$ **(B)** $17\frac{1}{2}$ **(C)** $17\frac{3}{4}$ **(D)** 31 **(E)** None of these

23. The graph of which of the following equations lies entirely above the x-axis?

(A) $y = 2x^2 - 6x + 6$ **(B)** $y = x^2 + 4x$ **(C)** $y = x^2 + 2x - 2$
(D) $y = -x^2 - 4x - 3$ **(E)** $y = -x^2$

24. Simplify $\dfrac{4x^{-2} - 2x^{-4}}{2x^{-3}}$.

(A) $2x - x^{-1}$ **(B)** $\dfrac{2x^3}{4x^2 - 2x^4}$ **(C)** $2x - 2x^{-4}$
(D) $4x^{-2} - x$ **(E)** $\dfrac{x}{2} - x^{-1}$

25. The value of the determinant $\begin{vmatrix} 2 & 4 & 0 \\ 8 & 6 & 0 \\ 3 & 2 & 1 \end{vmatrix}$ is

(A) -13 **(B)** -20 **(C)** 20 **(D)** 44 **(E)** None of these

26. If $3^{2x} + 9 = 10(3^x)$, then $x^2 - 2x + 1 =$

(A) 1 only **(B)** 5 only **(C)** 1 or 5 **(D)** 2 **(E)** 10

27. If $\log_x 3^{-2} + \log_x 27 = 2$, find x.

(A) 3^2 **(B)** 3^{-1} **(C)** $3^{1/2}$ **(D)** 3 **(E)** $\dfrac{1}{3}$

28. If $\log 2 = 0.3010$ and $\log 3 = 0.4771$, then $\log 135$ is approximately

(A) 2 **(B)** 2.11 **(C)** 2.13 **(D)** 2.15 **(E)** 1.73

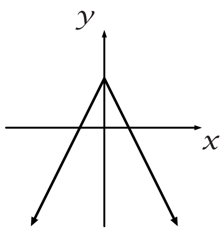
29. Find the maximum height of the curve $y = 12\cos x - 5\sin x$ above the x-axis.

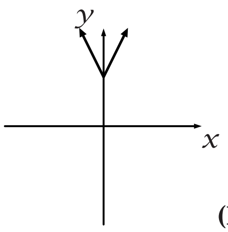
(A) 5 **(B)** 7 **(C)** 12 **(D)** 13 **(E)** 17

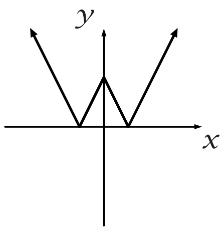
30. A person drives from town A to town B and averages 60 mph and on the return trip averages 40 mph. To make the round trip in the same amount of time at a fixed rate of speed, it would have to be

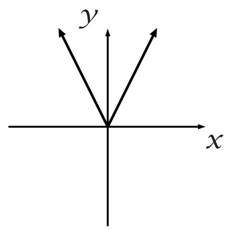
(A) 50 mph **(B)** 70 mph **(C)** 48 mph **(D)** 52 mph **(E)** None of these

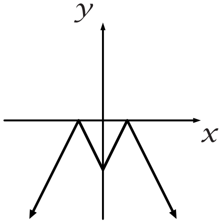
31. If the figure below shows the graph of $y = f(x)$, then which of the following is the graph of $y = |f(x)|$?



(A)


(B)


(C)


(D)


(E) None of these

32. Find the constant c if the graph of $y = x^2 + c$ is to be tangent to the line $y = x - 2$.

(A) -2 **(B)** $-\frac{7}{4}$ **(C)** $-\frac{3}{2}$ **(D)** $\frac{1}{2}$ **(E)** $-\frac{5}{4}$

33. Which of the following have the same graphs in the xy-plane?

I. $y(x+2) = x^2 - 4$ II. $y = \frac{x^2 - 4}{x + 2}$ III. $y = x - 2$

(A) I and II **(B)** I and III **(C)** II and III **(D)** I, II, and III
(E) None of these have the same graph

34. When $x^{153} + 1$ is divided by $x - 1$, the remainder is

(A) 1 **(B)** -1 **(C)** 0 **(D)** 2 **(E)** None of these

35. All the members of the family of parabolas defined by $y = 2x^2 + K$, for real number K, have

(A) the same vertex
(B) the same y-intercept
(C) the same axis of symmetry
(D) a common tangent
(E) the same x-intercepts

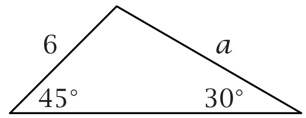
36. A coin is loaded such that a tail is three times as likely to occur as a head. The coin is flipped twice. Find the probability that two heads occur.

(A) $\frac{1}{9}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{16}$ **(D)** $\frac{1}{2}$ **(E)** None of these

37. The inverse function of $f(x) = \frac{x+1}{x}$ is

(A) x **(B)** $\frac{1}{x-1}$ **(C)** $x - 1$ **(D)** $x + 1$ **(E)** None of these

38. Given the triangle as shown, then $a = ?$



(A) $2\sqrt{6}$ **(B)** $3\sqrt{6}$ **(C)** $\frac{3\sqrt{2}}{2}$
(D) $3\sqrt{2}$ **(E)** $6\sqrt{2}$

39. Select the *incorrect* statement.

(A) $\sin(-x) = -\sin x$ **(B)** $\cos(-x) = \cos x$
(C) $\sin\left(\frac{\pi}{2} - x\right) = \cos x$ **(D)** $\sin(x + 2\pi) = \sin x$
(E) $\tan\left(\frac{\pi}{2} - x\right) = \tan x$

40. If \$1000 is deposited at ten percent compounded quarterly, after five years the amount on deposit in dollars would be

(A) $1000(1.10)^5$ **(B)** $1000(1.10)^{20}$ **(C)** $1000(1.025)^{20}$
(D) $1000(1.025)^5$ **(E)** None of these

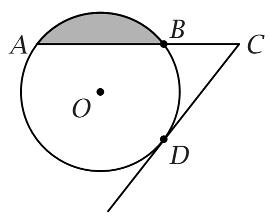
41. Factor completely: $x^2(y^3 + 1) - 2x(y^3 + 1) - 15(y^3 + 1)$.

(A) $(x + 5)(x - 3)(y + 1)(y^2 - 2y + 1)$
(B) $(x + 5)(x - 3)(y - 1)(y^2 + y + 1)$
(C) $(y + 1)(y^2 - y + 1)(x - 5)(x + 3)$
(D) $(y^3 + 1)(x - 5)(x + 3)$
(E) $(y^3 + 1)(x^2 - 2x - 15)$

42. If three dice labeled 1–6 are rolled, what is the probability that the sum of the top three numbers is 6?

(A) $\frac{1}{18}$ **(B)** $\frac{13}{216}$ **(C)** $\frac{5}{108}$ **(D)** $\frac{1}{12}$ **(E)** $\frac{1}{216}$

43. Circle O has radius 12, $BC = 4$, and $DC = 8$, and $\overline{CD}$ is tangent to circle O. Find the area of the shaded region.



(A) $24\pi - 72$ **(B)** $144\pi - 18\sqrt{3}$ **(C)** $\frac{14\pi - 18\sqrt{3}}{3}$
(D) $24\pi - 36\sqrt{3}$ **(E)** None of these

44. Find the shortest distance from the point $(\frac{9}{2}, 0)$ to the curve $y = \sqrt{x}$.

(A) $\frac{9}{2}$ **(B)** 2.25 **(C)** $\frac{3\sqrt{3}}{2}$ **(D)** $\frac{\sqrt{17}}{2}$ **(E)** $\frac{\sqrt{23}}{2}$

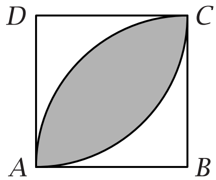
45. How many times does the graph of $f(x) = 3\cos(\frac{x}{2})$ intersect the x-axis for $0 \le x < 2\pi$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

46. The expression $\dfrac{2\cos 190^\circ + 2i\sin 190^\circ}{\cos 70^\circ + i\sin 70^\circ}$ is equal to

(A) 6 **(B)** $-1 + i\sqrt{3}$ **(C)** $\sqrt{3} - 1$ **(D)** $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$ **(E)** None of these

47. $ABCD$ is a square, with $AB = 4$. Point B is the center of the circle with $\overset{\frown}{AC}$. Point D is the center of the circle with $\overset{\frown}{AC}$. Find the area of the shaded region.



(A) $4(\pi - 2)$ **(B)** $8(\pi - 2)$ **(C)** $16(\pi - 2)$ **(D)** $8(\pi - 1)$ **(E)** $16\pi - 8$

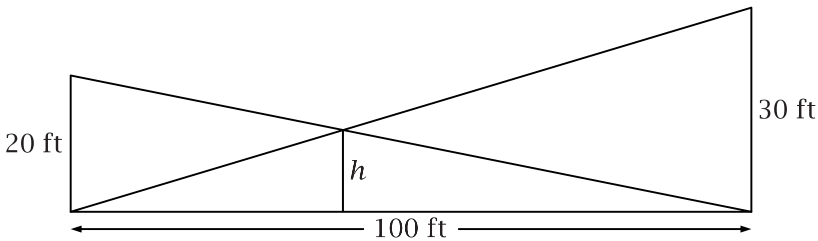
48. Evaluate $\cos^2 75^\circ - \sin^2 75^\circ$.

(A) 1 **(B)** $-\frac{1}{2}$ **(C)** -1 **(D)** $-\frac{\sqrt{3}}{2}$ **(E)** $\frac{\sqrt{3}}{2}$

49. How much corn worth \$2.20/kg should be added to oats worth \$2.80/kg to make 50 kg of grain feed worth \$2.40/kg?

(A) $33\frac{1}{3}$ kg **(B)** $16\frac{2}{3}$ kg **(C)** 25 kg **(D)** 4 kg **(E)** None of these

50. In the figure below, h equals



(A) 10 ft **(B)** 12 ft **(C)** 14 ft **(D)** 15 ft **(E)** None of these

The Ciphering

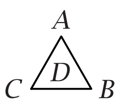
1. What is the area of a triangle whose sides are 13, 13, and 10?

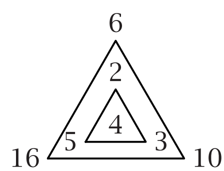
2. Find the distance from $(7, -2)$ to $2x + 3y - 7 = 0$.

3. Find the equation of the circle which passes through $(6, 0)$, $(0, 6)$, and $(6, 6)$.

4. Give the sum of all values of x such that $(x, 0)$ is on the graph of $f(x) = x^3 - x$.

5. A quadratic equation is written, with coefficients chosen at random from the numbers 1, 2, and 3, no number being used twice. What is the probability that the equation will have non-real complex roots?

6. If the symbol 

means $\dfrac{A \cdot B}{D - C}$, find 

.

7. Give the equations of all horizontal and vertical asymptotes of $y = \dfrac{x}{x^2 + x - 2}$.

8. If the sum of two numbers is 1 and their product is 1, then the sum of their squares is ?

9. If m and n are the roots of $x^2 + mx + n = 0$, $m \neq 0$, $n \neq 0$, then the value of the sum of the roots is ?

10. Compute $\dfrac{\sin 210^\circ}{\csc 210^\circ} + \dfrac{\cos 210^\circ}{\sec 210^\circ} + \tan 210^\circ \cot 210^\circ$.

2

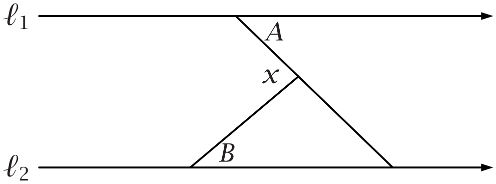
The 1983 State Tournament

The Written Test

1. Determine the domain of $y = \sqrt{\dfrac{2x+2}{x-3}}$.

(A) $\{x \mid x \geq -1 \text{ or } x \neq 3\}$ **(B)** $\{x \mid x > 3 \text{ or } x \leq -1\}$
(C) $\{x \mid x \neq 3\}$ **(D)** $\{x \mid x \geq 3\}$ **(E)** $\{x \mid x \leq 3\}$

2. If $\ell_1 \parallel \ell_2$, $m\angle A = 44^\circ$, $m\angle B = 40^\circ$, then $m\angle x =$



(A) 136° **(B)** 84° **(C)** 96° **(D)** 44° **(E)** 140°

3. Find the positive numerical value, in simplest form, of the expression

$$\sqrt{12+\sqrt{12+\sqrt{12+\sqrt{12+\cdots}}}}.$$

(A) 4 **(B)** 3.5 **(C)** 3 **(D)** 12 **(E)** 13

4. If r and s are the roots of $ax^2 + bx + c = 0$, the value of $\dfrac{1}{r^2} + \dfrac{1}{s^2}$ is

(A) $b^2 - 4ac$ **(B)** $\dfrac{b^2 - 4ac}{2a}$ **(C)** $\dfrac{b^2 - 4ac}{c^2}$ **(D)** $\dfrac{b^2 - 2ac}{c^2}$ **(E)** None of these

5. Find the area of the plane figure determined by the inequality $\frac{1}{2}|x| + |y| \leq 2$.

(A) 2 **(B)** 4 **(C)** 8 **(D)** 16 **(E)** None of these

6. The value of k that yields the same remainder when $x^3 + kx^2 + 2x + 7$ is divided by either $x + 1$ or $x + 2$ is equal to

(A) 3 **(B)** 4 **(C)** 1 **(D)** 0 **(E)** -4

7. The measure of an acute angle of a right triangle is twice the measure of the other. The measure of the longer leg is 6. What is the measure of the hypotenuse?

(A) $2\sqrt{3}$ **(B)** $4\sqrt{3}$ **(C)** $6\sqrt{3}$ **(D)** 10 **(E)** None of these

8. Three men working 3 hours per day for 3 days produce 3 kilograms of a certain substance. Assuming all men involved work with the same constant efficiency, how many kilograms of the substance will 4 men produce working 4 hours per day for 4 days?

(A) 4 **(B)** $\dfrac{4}{3}$ **(C)** $\dfrac{4}{9}$ **(D)** $\dfrac{64}{9}$ **(E)** None of these

9. In an arithmetic progression of positive numbers, the common difference is twice the first term, and the sum of the first six terms is equal to the square of the first term. Find the first term.

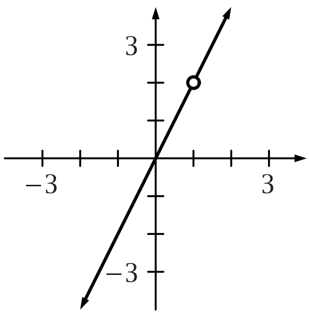
(A) 0 only **(B)** 6 only **(C)** 0 and 36 **(D)** 0 and 6 **(E)** 36 only

10. Find the equation of a line through the point $(2, 5)$ which makes an angle of 45° with the line $x - 3y + 6 = 0$.

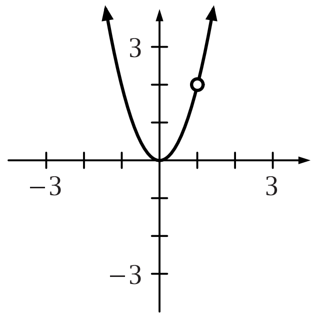
(A) $2x - y + 1 = 0$ **(B)** $x - y + 3 = 0$ **(C)** $3x - y - 1 = 0$ **(D)** $x - 3y + 13 = 0$ **(E)** $x + 3y - 17 = 0$

11. Which of the following figures best represents the graph of f defined below?

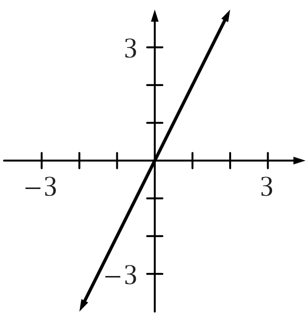
$$f(x) = \begin{cases} 2x - 1 & x < 1 \\ 2 & x = 1 \\ 2x^2 & x > 1 \end{cases}$$



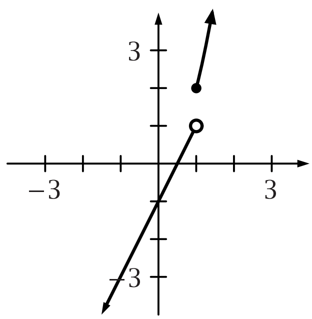
(A)



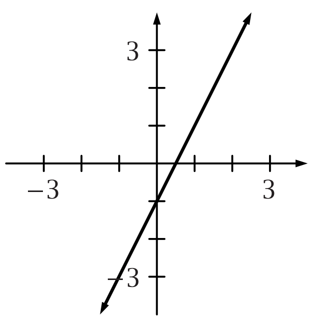
(B)



(C)



(D)



(E)

12. How many numbers greater than 1000 can be formed with the digits 2, 3, 0, and 4 if no digit is repeated?

(A) 3 **(B)** 4 **(C)** 18 **(D)** 24 **(E)** 256

13. What is the largest counting number which will divide the product of any four consecutive integers?

(A) 24 **(B)** 16 **(C)** 8 **(D)** 4 **(E)** 2

14. A basketball goal has the equation $x^2 + y^2 + 4x - 2y = 0$, and a ball with a radius of 1 comes down with its center at the point of intersection of $7x - 4y = -15$ and $2x + 3y = 4$. What happens to the ball?

(A) Scores 2 points without touching the rim.
(B) Scores 2 points but touches the rim.
(C) Touches the rim but doesn't go through the hoop.
(D) Misses the hoop entirely.
(E) The ball is too big to go through the hoop.

15. Find the value of $\lim_{h \to 0} \dfrac{\sqrt{4+h} - \sqrt{4}}{h}$.

(A) 1 **(B)** $\frac{1}{2}$ **(C)** $\frac{1}{4}$ **(D)** 2 **(E)** None of these

16. A box contains 5 black and 3 white balls. Three balls are drawn in succession. What is the probability of drawing 2 black balls and 1 white ball?

(A) $\frac{3}{8}$ **(B)** $\frac{5}{28}$ **(C)** $\frac{2}{15}$ **(D)** $\frac{15}{28}$ **(E)** $\frac{5}{8}$

17. Assume that the number of bacteria, in thousands, present in a certain culture at time t is given by $N(t) = 24t(10 - 2t) + 15$. Find the maximum number of bacteria (in thousands).

(A) 24 **(B)** 240 **(C)** 250 **(D)** 315 **(E)** 500

18. Suppose a ball is thrown directly upward with a speed of 96 ft/sec and moves according to the law $y = 96t - 16t^2$, where y is the height in feet above the starting point, and t is the time in seconds after it is thrown. What is the greatest height, in feet, reached by the ball?

(A) 96 **(B)** 80 **(C)** 112 **(D)** 144 **(E)** 128

19. Find two numbers whose sum is 8 such that the sum of the squares of the two numbers is minimized.

(A) 2, 6 **(B)** $4+\sqrt{2}, 4-\sqrt{2}$ **(C)** 3, 5 **(D)** 4, 4 **(E)** $3+\sqrt{3}, 5-\sqrt{3}$

20. The percent that m is greater than n is

(A) $\dfrac{100(m-n)}{m}$ **(B)** $\dfrac{100(m-n)}{n}$ **(C)** $\dfrac{m-n}{n}$ **(D)** $\dfrac{m-n}{m}$ **(E)** $\dfrac{100(m+n)}{n}$

21. For what values of k will $\left\langle \dfrac{k-2}{13}, \dfrac{k+5}{13} \right\rangle$ be a unit vector?

(A) 7, 15 **(B)** −10, 7 **(C)** −5, 15 **(D)** 2, 7 **(E)** 7 only

22. The total number of diagonals in an octagon is

(A) 28 **(B)** 20 **(C)** 15 **(D)** 24 **(E)** None of these

23. If $F(n+1) = \dfrac{4F(n)+1}{4}$ for $n = 1, 2, 3, \ldots$, and $F(101) = 27$, then $F(1) =$

(A) −81 **(B)** −2 **(C)** 0 **(D)** 2 **(E)** 3

24. Find the coordinates of the point $P(x, y)$ which divides the line segment from $P_1(1, 7)$ to $P_2(6, -3)$ in the ratio 2 : 3.

(A) $\left(\dfrac{14}{3}, \dfrac{8}{3}\right)$ **(B)** $\left(\dfrac{10}{3}, \dfrac{20}{3}\right)$ **(C)** $(5, 0)$ **(D)** $(3, 3)$ **(E)** None of these

25. If $\sin 2x = -\dfrac{1}{2}$ and $\cos 2x = -\dfrac{\sqrt{3}}{2}$, then $x =$

(A) 105° **(B)** 210° **(C)** 240° **(D)** 120° **(E)** Cannot be determined

26. What is the value printed by the program below?

```
10 LET A = 1
20 LET B = 2
30 FOR N = 1 TO 5
40 LET A = B * A
50 LET B = A / B
60 NEXT N
70 PRINT B
80 END
```

(A) 2 **(B)** 4 **(C)** 8 **(D)** 16 **(E)** 32

27. Find an equation of a line parallel to the line $12x - 5y - 15 = 0$ and at a perpendicular distance from it equal to k.

(A) $12x - 5y - 15 - k = 0$ **(B)** $12x - 5y - 15 + k = 0$ **(C)** $12x - 5y + 13k - 15 = 0$ **(D)** $12x - 5y + 2k = 0$ **(E)** $12x - 5y + 2k - 15 = 0$

28. The graph of $x^2 - x + xy + y - 2y^2 = 0$ is what?

(A) a hyperbola **(B)** an ellipse **(C)** a parabola **(D)** 2 intersecting parabolas **(E)** 2 intersecting lines

29. The angle between two lines L_1 and L_2 is 45°. If the slope of L_1 is $\frac{2}{3}$, determine a slope of L_2.

(A) $-\dfrac{3}{2}$ **(B)** $-\dfrac{2}{3}$ **(C)** $\dfrac{3}{2}$ **(D)** 5 **(E)** -5

30. How many points do the graphs of the equations $x^2 + y^2 = 25$ and $y = x^2$ have in common?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

31. Find the integer x given that

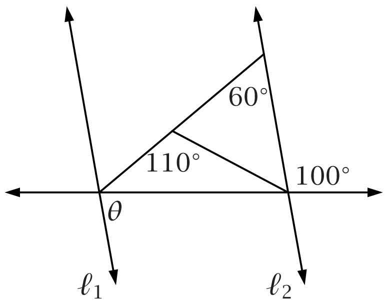
$$x = \log\frac{2}{1} + \log\frac{3}{2} + \log\frac{4}{3} + \cdots + \log\frac{100}{99}.$$

(A) -2 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** None of these

32. The base of a rectangular solid is a rectangle whose sides are 2 and 6. If the diagonal of the solid is 11, find its volume.

(A) $12\sqrt{161}$ **(B)** 36 **(C)** 73 **(D)** 108 **(E)** 132

33. Given the figure with ℓ_1 parallel to ℓ_2 and the indicated angle measures, find the measure of $\angle\theta$.



(A) 40° **(B)** 50° **(C)** 60° **(D)** 70° **(E)** 80°

34. Which of the following is equal to $\sqrt[3]{x\sqrt[4]{x\sqrt[3]{x}}}$?

(A) $\sqrt[3]{x^2}$ **(B)** $\sqrt[9]{x^4}$ **(C)** $\sqrt[9]{x^6}$ **(D)** $\sqrt[36]{x}$ **(E)** None of these

35. Which of the following sets contains a number that is not in the range of $g(x) = \dfrac{2x+11}{3x-7}$?

(A) $\left\{\frac{1}{2}, 7, 2, -\frac{3}{5}, 6\right\}$ **(B)** $\left\{\frac{3}{4}, \frac{5}{9}, -\frac{7}{11}, -\frac{11}{7}\right\}$

(C) $\left\{1, \frac{7}{3}, -4, -11, -7\right\}$ **(D)** $\left\{\frac{2}{3}, \frac{5}{4}, -17, \frac{2}{7}\right\}$

(E) $\left\{\frac{11}{2}, 7, 0, \frac{3}{7}\right\}$

36. Find the area of the region that lies inside the graph of $x^2 + y^2 - 4x - 6y - 12 = 0$ and that is outside the graph of $x^2 + y^2 - 2x - 4y + 4 = 0$.

(A) 12π **(B)** 24π **(C)** 29π **(D)** 30π **(E)** None of these

37. If $\sin\alpha = \frac{1}{3}$ and $\tan\beta = 1$, with α and β in the first quadrant, then $\cos(\alpha+\beta)$ equals

(A) $\dfrac{\sqrt{2}+4}{3}$ **(B)** $\dfrac{4-\sqrt{2}}{6}$ **(C)** $\dfrac{\sqrt{2}-4}{6}$ **(D)** $\dfrac{\sqrt{2}+4}{6}$ **(E)** $\dfrac{7\sqrt{2}}{6}$

38. Let

$$\frac{5x+1}{x-x^3} = \frac{A}{x} + \frac{B}{x+1} - \frac{C}{x-1}$$

be an identity in x. The numerical value of $A+B+C$ is

(A) 6 **(B)** 0 **(C)** -6 **(D)** 4 **(E)** 5

39. If $V = \pi r^2 h$, then $\log r$ equals

(A) $\dfrac{\log V - \log\pi - \log h}{2}$ **(B)** $\sqrt{\log V - \log\pi - \log h}$
(C) $\sqrt{\log\pi + \log h - \log V}$ **(D)** $\dfrac{\log\pi + \log h - \log V}{2}$
(E) $\dfrac{\pi}{2}(\log V - \log h)$

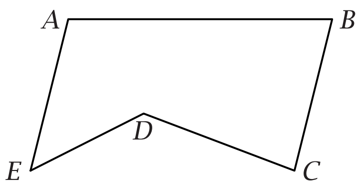
40. Write an equation of a line making an angle of 120° with the positive horizontal axis and a distance of 5 units from the origin.

(A) $\sqrt{3}x + y + 5 = 0$ **(B)** $\sqrt{3}x - y + 5 = 0$ **(C)** $\sqrt{3}x - y - 5 = 0$
(D) $\sqrt{3}x - 3y = 0$ **(E)** $\sqrt{3}x + y - 10 = 0$

41. Solve $\log_4 x - \dfrac{1}{\log_4 x} = \dfrac{3}{2}$.

(A) $2\sqrt{2}$ **(B)** $-\dfrac{1}{2}, 2$ **(C)** $\dfrac{1}{2}, 16$ **(D)** 2 **(E)** None of these

42. If $\overline{AE} \parallel \overline{BC}$, find the sum of the measures of the angles in the figure given.



(A) 360° **(B)** 480° **(C)** 540° **(D)** 620°
(E) Cannot be determined from the given information

43. Determine the angle through which the axes must be rotated to remove the xy term in the equation $7x^2 - 6\sqrt{3}xy + 13y^2 = 16$.

(A) $90°$ **(B)** $60°$ **(C)** $45°$ **(D)** $30°$ **(E)** None of these

44. Find the area of the pentagon whose vertices are $(-5,-2)$, $(-2,5)$, $(2,7)$, $(5,1)$, and $(2,-4)$.

(A) $2\sqrt{33}$ **(B)** 66 **(C)** $\frac{\sqrt{58}}{2}$ **(D)** $\sqrt{58}$ **(E)** None of these

45. The common difference of the arithmetic sequence whose second term is 4 and whose ninth term is -17 is

(A) $-6\frac{1}{2}$ **(B)** -3 **(C)** $6\frac{1}{2}$ **(D)** 7 **(E)** -13

46. The arithmetic mean of $\frac{1}{2}$ and $\frac{25}{2}$ exceeds the positive geometric mean by

(A) 2 **(B)** 4 **(C)** 8 **(D)** 9 **(E)** 11

47. The length ℓ of a tangent, drawn from a point A to a circle, is $\frac{4}{3}$ of the radius r. The shortest distance from A to the circle is

(A) $\frac{r}{2}$ **(B)** r **(C)** $\frac{\ell}{2}$ **(D)** $\frac{2\ell}{3}$ **(E)** a value between r and ℓ

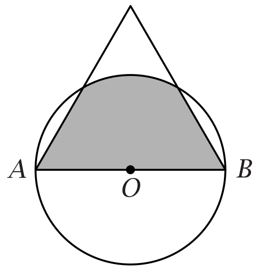
48. In how many different ways can the letters of the word `ATLANTA` be arranged?

(A) 24 **(B)** 420 **(C)** 1540 **(D)** 4620 **(E)** 9240

49. A bowl contains 20 slips of paper numbered 1 to 20 consecutively. If a slip is randomly drawn, what is the probability of drawing a prime number?

(A) $\frac{9}{20}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{11}{20}$ **(D)** $\frac{3}{10}$ **(E)** None of these

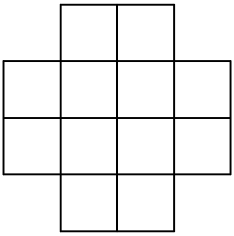
50. Circle O with diameter $\overline{AB}$ and equilateral triangle ABC is given in the figure. If the area of circle O is 4π, find the perimeter of the shaded region.



(A) $\frac{\pi}{3} + 4$ **(B)** $\frac{2\pi}{3} + 4$ **(C)** $\frac{4\pi}{3} + 8$ **(D)** $\frac{\pi}{8} + 3$ **(E)** $\frac{2\pi}{3} + 8$

The Ciphering

1. How many squares are there in the diagram below?

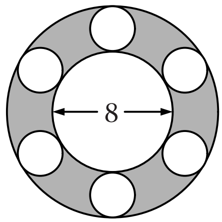


2. Find all values of a for which

$$\frac{\dfrac{a}{a+1} + \dfrac{a}{a-1}}{\dfrac{a}{a+1} - \dfrac{a}{a-1}}$$

is undefined.

3. Find the area, in square inches, of the shaded region if the smaller circles have diameters of 3 inches.



4. Find the value of $\cos\left(\text{Arcsin}\,\frac{4}{5} + \text{Arctan}\,\frac{5}{12}\right)$.

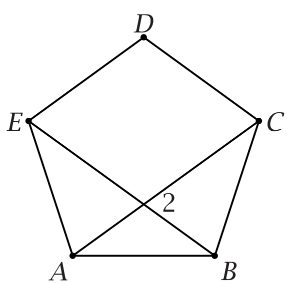
5. Wonder Woman, Batman, and Superman are in a race. Wonder Woman is three times as likely to win the race as Batman, and Batman is twice as likely to win the race as Superman. Find the probability that Superman wins the race.

6. A function f, defined on the real numbers, satisfies $f(x+y) = f(x) + f(y)$. If $f(8) = 2$, find the value of $f(4)$.

7. Write the complex fraction below as a single fraction in simplest form.

$$\frac{1 - \dfrac{1}{\dfrac{a}{b} + 2}}{1 + \dfrac{3}{\dfrac{a}{2b} + 1}}$$

8. $ABCDE$ is a regular pentagon, where $\overline{AC}$ and $\overline{BE}$ are diagonals. The measure of $\angle 2$ = ?



9. The year 1849 was a square. What is the next year that was a square?

10. Find the value of x which satisfies $\log_2(\log_3(\log_4 x)) = 0$.

Part II

The Solutions

3

Solutions to the 1982 State Tournament

The Written Test Solutions

1. **D** We have $-5 \le x - 4 \le 5$, and upon adding 4 to everything, we get $-1 \le x \le 9$.

2. **B** A discount of $p\%$ is equivalent to multiplication by $1 - \frac{p}{100}$. Hence, the discounts of 5%, 10%, and 20% imply that a price will be multiplied by $\frac{19}{20}$, $\frac{9}{10}$, and $\frac{4}{5}$. This is a total factor of $\frac{19}{20} \cdot \frac{9}{10} \cdot \frac{4}{5} = \frac{19 \cdot 9}{5 \cdot 10 \cdot 5} = \frac{171}{250}$. This is a discount of $1 - \frac{171}{250} = \frac{250-171}{250} = \frac{79}{250} = \frac{316}{1000} = 31.6\%$.

3. **D** The curved surface is a rectangle which has height 4 and width equal to half the circumference of the cylinder; this width is $\frac{1}{2} \cdot 4\pi = 2\pi$. The shortest distance is then the diagonal of the rectangle: $\sqrt{4^2 + (2\pi)^2} = 2\sqrt{4 + \pi^2}$.

4. **A** John types $\frac{1}{15}$ lists per hour and Mary types $\frac{1}{10}$ lists per hour, so together, they can type $\frac{1}{10} + \frac{1}{15} = \frac{25}{150} = \frac{1}{6}$ per hour. Hence, it will take them 6 hours to type the entire list.

5. **D** The circumference is $2\pi r = x$ feet so the radius is $r = \frac{x}{2\pi}$ feet. Converting to yards, the radius is $r = \frac{x}{6\pi}$ yards. Thus, the area is $A = \pi r^2 = \pi(\frac{x}{6\pi})^2 = \frac{x^2}{36\pi}$ square yards.

6. **B** The bullet covered a distance of x feet at 2000 ft/sec, and the sound covered a distance of x feet at 1100 ft/sec. The bullet

and the sound took 6.2 seconds. Thus,

$$\begin{aligned}
6.2 &= \frac{x}{2000} + \frac{x}{1100} \\
620 &= \frac{x}{20} + \frac{x}{11} \\
620 &= \frac{11x + 20x}{220} \\
x &= \frac{620 \cdot 220}{31} = 2 \cdot 2200 = 4400.
\end{aligned}$$

7. **A** Since $9^t = (3^2)^t = 3^{2t}$, we have $3^{3t} = 3^{2t}$ so that $3t = 2t$. The only solution is $t = 0$.

8. **D** As x increases without bound, the function approaches $\frac{x}{x} = 1$. So the horizontal asymptote is $y = 1$. Since $x = -4$ makes the denominator zero, the vertical asymptote is $x = -4$.

9. **D** We have

$$\begin{aligned}
(x+y)^{-1}(x^{-1} + y^{-1}) &= \frac{1}{x+y}\left(\frac{1}{x} + \frac{1}{y}\right) \\
&= \frac{1}{x+y}\left(\frac{y+x}{xy}\right) \\
&= \frac{1}{xy}.
\end{aligned}$$

10. **D** We use the determinant method. We have

$$\begin{aligned}
\frac{1}{2}\begin{vmatrix} 0 & 1 & 3 & 0 \\ 1 & 4 & 6 & 1 \end{vmatrix} &= \frac{1}{2}|0 + 6 + 3 - (1 + 12 + 0)| \\
&= \frac{1}{2} \cdot |-4| = 2.
\end{aligned}$$

11. **B** With the equation in this form, it is easy to see that the y-intercept is $(0, 4)$ and the x-intercept is $(3, 0)$. Hence the slope is $-\frac{4}{3}$.

12. **E** We have $\sqrt[n]{3^{n+2} \div 9} = \sqrt[n]{\frac{3^{n+2}}{3^2}} = \sqrt[n]{3^n} = 3.$

13. **C** One can solve this in multiple ways. We present three ways.

First Solution. Rationalize the numerator. This yields

$$\lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} = \lim_{h\to 0} \frac{4 + h - 4}{h(\sqrt{4+h} + 2)} = \lim_{h\to 0} \frac{1}{\sqrt{4+h} + 2} = \frac{1}{4}.$$

SECOND SOLUTION. Recognize that this is the definition of the derivative of the function $f(x) = \sqrt{x}$ at the point where $x = 4$. Since $f'(x) = \frac{1}{2\sqrt{x}}$, $f'(4) = \frac{1}{4}$.

THIRD SOLUTION. Since both numerator and denominator approach zero as $h \to 0$, we may use l'Hôpital's rule with the variable h. This gives us

$$\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} = \lim_{h\to 0} \frac{\frac{1}{2}(4+h)^{-1/2}}{1} = \lim_{h\to 0} \frac{1}{2\sqrt{4+h}} = \frac{1}{4}.$$

14. **A** Drop a perpendicular from A to $\overline{BE}$ at F. Then, by symmetry, $AF = 4$ and $BF = 3$. Hence $EF = 3 + 4 = 7$, and by the Pythagorean Theorem, we have $AE = \sqrt{AF^2 + FE^2} = \sqrt{16 + 49} = \sqrt{65}$.

15. **C** Half the base has length x, so the height of the triangle has length $x\sqrt{3}$. Hence $y = x\sqrt{3}$.

16. **E** We trace through the program in the table below.

J	B	A
	1	1
1	$1 + 1 = 2$	$1 + 2 = 3$
2	$2 + 3 = 5$	$3 + 5 = 8$
3	$5 + 8 = 13$	$8 + 13 = 21$
4	$13 + 21 = 34$	$21 + 34 = 55$
5	$34 + 55 = 89$	$55 + 89 = 144$
6	$89 + 144 = 233$	$144 + 233 = 377$
7	$233 + 377 = 610$	$377 + 610 = 987$

The number printed is therefore 987.

17. **B** The first student can choose any of the seven seats; the second student can choose any of the six remaining seats; the third student can choose five; and the fourth student can choose four. This is $7 \cdot 6 \cdot 5 \cdot 4$ which is equivalent to 7! divided by 3!. This is a permutation of four objects from a set of seven objects: $P(7,4)$.

18. **C** The diagonal of a rectangular solid of length ℓ, width w, and height h is $\sqrt{\ell^2 + w^2 + h^2}$. Hence, $14 = \sqrt{4^2 + 6^2 + h^2}$. This implies $14^2 = 196 = 16 + 36 + h^2$ so that $h^2 = 144$ whence $h = 12$. The volume is then $12 \cdot 6 \cdot 4 = 288$.

19. **A** The transformation is a translation five units to the left and 10 units up, which leaves the area unchanged.

20. **A** Since $f(-2) = |-2-1| = 3$ and $g(-3) = 1-(-3)^2 = -8$, we have $3f(-2) + 4g(-3) = 3 \cdot 3 + 4(-8) = 9 - 32 = -23$.

21. **B** We have

$$\begin{aligned} 3(2x+7)-5 &= 2(5x-4)+4x \\ 6x+21-5 &= 10x-8+4x \\ 6x+16 &= 14x-8 \\ 24 &= 8x \\ x &= 3. \end{aligned}$$

22. **B** There is a common difference of $\frac{1}{4}$, and the first term is 2, so the nth term is given by $2 + \frac{1}{4}(n-1)$. When $n = 63$, we get $2 + \frac{62}{4} = 2 + 15\frac{1}{2} = 17\frac{1}{2}$.

23. **A** The graph of a quadratic lies entirely above the x-axis when it has no real roots; it has no real roots when the discriminant is negative. So we compute the discriminant of each of the answer choices. Starting with choice A, we find that $(-6)^2 - 4 \cdot 2 \cdot 6 = 36 - 48 < 0$ and this must be the answer.

24. **A** Multiplying numerator and denominator by x^4 gives us

$$\frac{4x^2-2}{2x} = \frac{2x^2-1}{x} = 2x - \frac{1}{x} = 2x - x^{-1}.$$

25. **B** We calculate the determinant.

$$\begin{aligned} \begin{vmatrix} 2 & 4 & 0 \\ 8 & 6 & 0 \\ 3 & 2 & 1 \end{vmatrix} &= 2\begin{vmatrix} 6 & 0 \\ 2 & 1 \end{vmatrix} + 4\begin{vmatrix} 8 & 0 \\ 3 & 1 \end{vmatrix} \\ &= 2(6-0) - 4(8-0) = 12 - 32 = -20. \end{aligned}$$

26. **A** Note that the equation may be rewritten as $(3^x)^2 - 10(3^x) + 9 = (3^x - 1)(3^x - 9) = 0$ so that $3^x = 1$ or $3^x = 9$. Hence, $x = 0$ or $x = 2$. Then $x^2 - 2x + 1 = (x-1)^2 = 1$ if $x = 0$ or if $x = 2$.

27. **C** Combining logarithms, we have $\log_x(3^{-2} \cdot 27) = \log_x 3 = 2$. Thus, $3 = x^2$, whence $x = \sqrt{3} = 3^{1/2}$. (Note we reject the negative roots since we cannot have a negative base.)

28. **C** Since $135 = 5 \cdot 3^3$, we see that $\log 135$ can be written as $\log 135 = \log 5 + 3\log 3$. Since $5 = \frac{10}{2}$, we can write $\log 135 = \log 10 - \log 2 + 3\log 3 = 1 - \log 2 + 3\log 3 = 1 - 0.3010 + 3 \cdot 0.4771 = 0.6990 + 1.4313 = 2.1303$.

29. **D** The maximum occurs when the derivative is zero; hence $y' = -12\sin x - 5\cos x = 0$ implies $\tan x = -\frac{5}{12}$. At this value of x, we have $\sin x = -\frac{5}{13}$ and $\cos x = \frac{12}{13}$. Thus, $y = 12\cos x - 5\sin x = 12 \cdot \frac{12}{13} + 5 \cdot \frac{5}{13} = \frac{144+25}{13} = \frac{169}{13} = 13$.

30. **C** Average speed is the total distance divided by the total time. If we let the distance from A to B be d, then the total distance is $2d$ and the total time is $\frac{d}{60} + \frac{d}{40}$. Thus, the average speed is

$$\frac{2d}{\frac{d}{60} + \frac{d}{40}} = \frac{2}{\frac{1}{60} + \frac{1}{40}} = \frac{2}{\frac{100}{60 \cdot 40}} = \frac{2 \cdot 60 \cdot 40}{100} = 48.$$

31. **B** Taking the absolute value of a function keeps any part of the graph lying above x-axis where it is, and reflects about the x-axis any part below the x-axis. This must be the graph in B.

32. **B** The slope of the line is 1, so we set the derivative of $y = x^2 + c$ equal to 1: $y' = 2x = 1$ which implies $x = \frac{1}{2}$. When $x = \frac{1}{2}$, then $y = x - 2 = \frac{1}{2} - 2 = -\frac{3}{2}$. Hence, the number c is given by $c = y - x^2 = -\frac{3}{2} - (\frac{1}{2})^2 = -\frac{3}{2} - \frac{1}{4} = -\frac{7}{4}$.

33. **E** Note that the critical value which could make the graphs distinct is $x = -2$. When $x = -2$, the equation $y(x+2) = x^2 - 4$ reduces $y \cdot 0 = 0$ so that y could be any real number whatsoever; its graph consists of two intersecting lines, $y = x - 2$ and $x = -2$. When $x = -2$, the equation $y = \frac{x^2-4}{x+2}$ is undefined; its graph is the graph of $y = x - 2$ with a hole where $x = -2$. Finally, the graph of $y = x - 2$ is a continuous line. So none of these graphs would be the same.

34. **D** Let $f(x) = x^{153} + 1$. By the Remainder Theorem, the remainder upon dividing $f(x)$ by $x - 1$ is $f(1) = 1^{153} + 1 = 2$.

35. **C** Adding K to $y = 2x^2$ simply results in a vertical shift. This means the possible y-coordinates of any common x-coordinate are different. Hence, the family of parabolas cannot have the same vertex, intercepts, or tangents. The only possibility is that they have the same axis of symmetry, $x = 0$.

36. **C** If a tail is three times as likely to occur as a head, then the probability of flipping tails is $\frac{3}{4}$ and heads is $\frac{1}{4}$. Thus, the probability that we get two heads is $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$.

37. **B** To find the inverse of $y = f(x)$, we interchange y and x and solve for y. This yeilds

$$\begin{aligned} x &= \frac{y+1}{y} \\ xy &= y+1 \\ xy - y &= 1 \\ y &= \frac{1}{x-1}. \end{aligned}$$

38. **E** We use the Law of Sines. We have

$$\frac{\sin 30^\circ}{6} = \frac{\sin 45^\circ}{a}.$$

Then

$$a = \frac{6\sin 45^\circ}{\sin 30^\circ} = \frac{6 \cdot \frac{\sqrt{2}}{2}}{\frac{1}{2}} = 6\sqrt{2}.$$

39. **E** We can use a single value of x to act as a counterexample in order to identify the incorrect statement. Using $x = \frac{\pi}{2}$ reveals that statement E cannot be true.

40. **C** Interest at ten percent compounded quarterly implies that in each quarter year, the deposit earns $\frac{10\%}{4} = 2.5\%$. Hence, the total amount in the account is $1000(1+0.025)^{5\cdot 4} = 1000(1.025)^{20}$.

41. **C** We see that the expression factors as

$$\begin{aligned} (y^3+1)(x^2-2x-15) &= (y^3+1)(x-5)(x+3) \\ &= (y+1)(y^2-y+1)(x-5)(x+3). \end{aligned}$$

42. **C** A sum of six can be obtained by rolling 1, 1, 4; 1, 2, 3; or 2, 2, 2. There are 3 ways to roll 1, 1, 4. There are 6 ways to roll 1, 2, 3. There is only 1 way to roll 2, 2, 2. The total number of ways to obtain 6 is therefore $3 + 6 + 1 = 10$ out of $6^3 = 216$ ways to roll three dice. The probability is therefore $\frac{10}{216} = \frac{5}{108}$.

43. **D** By Power of a Point, we have $AC \cdot BC = CD^2$. Then $4AC = 64$ so that $AC = 16$. Thus $AB = AC - BC = 16 - 4 = 12$. It follows that AB is the same length as the radius of the circle. Hence, $\triangle AOB$ is equilateral, and its area is $\frac{\sqrt{3}}{4} \cdot 12^2 = 36\sqrt{3}$. The area of sector AOB is one-sxith of the area of the circle: $\frac{1}{6} \cdot \pi \cdot 12^2 = 24\pi$. Finally, the area of the shaded segment of the circle is $24\pi - 36\sqrt{3}$.

44. **D** We want to find the minimum distance from $(\frac{9}{2}, 0)$ to a point (x, y) on the graph of $y = \sqrt{x}$. This distance D is given by $D^2 = (x-\frac{9}{2})^2 + (y-0)^2 = (x-\frac{9}{2})^2 + (\sqrt{x}-0)^2 = x^2 - 9x + \frac{81}{4} + x = x^2 - 8x + \frac{81}{4}$. The distance is minimized when the square of the distance is minimized; the square of the distance is a quadratic, so the minimum occurs at the vertex. The x-coordinate of the vertex of $x^2 - 8x + \frac{81}{4}$ is $-\frac{b}{2a} = -\frac{-8}{2} = 4$. Hence the minimum square of the distance is $D^2 = 4^2 - 8 \cdot 4 + \frac{81}{4} = \frac{81}{4} - 16 = \frac{81-64}{4} = \frac{17}{4}$. Finally, the shortest distance is $\frac{\sqrt{17}}{2}$.

45. **B** We are really asking for how many solutions are there to the equation $3\cos(\frac{x}{2}) = 0$ for $0 \le x < 2\pi$. There are two angles where cosine is zero on the unit circle, and since cut that angle in half, there is only $\frac{2}{2} = 1$ solution to this equation.

46. **B** Expressing the numbers in exponential form, we have

$$\frac{2\cos 190^\circ + 2i\sin 190^\circ}{\cos 70^\circ + i\sin 70^\circ} = \frac{2e^{190^\circ i}}{e^{70^\circ i}} = 2e^{(190^\circ - 70^\circ)i} = 2e^{120^\circ i}$$

which is equivalent to $2\cos 120^\circ + 2i\sin 120^\circ = -1 + i\sqrt{3}$.

47. **B** Split the shaded region along the diagonal $\overline{AC}$. Then half of this region is the segment of a circle of radius 4, formed from a chord of length $4\sqrt{2}$ and angle 90°. Hence the area of the segment is $\frac{1}{4} \cdot \pi \cdot 4^2 - \frac{1}{2} \cdot 4 \cdot 4 = 4\pi - 8$. The shaded region is twice this: $2(4\pi - 8) = 8(\pi - 2)$.

48. **D** We use the cosine double-angle identity:

$$\cos^2 75^\circ - \sin^2 75^\circ = \cos(2 \cdot 75^\circ) = \cos 150^\circ = -\frac{\sqrt{3}}{2}.$$

49. **A** Let c and o be the kg of corn and oats to use. Then $c + o = 50$ and

$$\begin{aligned} \$2.40 &= \frac{\$2.20c + \$2.80o}{c + o} = \frac{\$2.20c + \$2.80o}{50} \\ \$120 &= \$2.20c + \$2.80o \\ 1200 &= 22c + 28o \\ 600 &= 11c + 14o. \end{aligned}$$

Multiplying the equation $c + o = 50$ by 14 and subtracting the above equation yields $100 = 3c$ so that $c = 33\frac{1}{3}$.

50. **B** The segment h splits the segment of length 100 into two pieces; call the shorter piece x and the longer $100 - x$. Then we have similar triangles in the diagram which give us

$$\frac{h}{100 - x} = \frac{20}{100} = \frac{1}{5} \qquad \text{and} \qquad \frac{h}{x} = \frac{30}{100} = \frac{3}{10}.$$

The second proportion yields $x = \frac{10h}{3}$. Using this in the first proportion, we obtain

$$h = \frac{100 - x}{5} = \frac{100 - \frac{10h}{3}}{5} = \frac{300 - 10h}{15} = 20 - \frac{2}{3}h.$$

It follows that $h = 12$.

The Ciphering Solutions

1. **60** The triangle is isosceles, and an altitude dropped from the vertex angle bisects and is perpendicular to the side of length 10. By the Pythagorean Theorem, the altitude is $\sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12$. Hence the area is $\frac{1}{2} \cdot 12 \cdot 10 = 60$.

2. $\mathbf{\sqrt{13}/13}$ We have

$$\frac{|2 \cdot 7 - 3 \cdot 2 - 7|}{\sqrt{2^2 + 3^2}} = \frac{|14 - 6 - 7|}{\sqrt{4 + 9}} = \frac{1}{\sqrt{13}} = \frac{\sqrt{13}}{13}.$$

3. $\mathbf{(x{-}3)^2 + (y{-}3)^2 = 18}$ Note that the points create a right triangle with hypotenuse from $(0, 6)$ to $(6, 0)$. This hypotenuse has length $6\sqrt{2}$ and is the diameter of the circle. It follows that the center of the circle is $(3, 3)$ and the radius is $3\sqrt{2}$; thus the equation is $(x - 3)^2 + (y - 3)^2 = 18$.

4. **0** By Viéte's Formulas, the sum is 0.

5. **2/3** Call the coefficients a, b, and c. There are $3! = 6$ ways to choose these values from 1, 2, and 3. We want non-real complex roots, so we require that $b^2 - 4ac < 0$. Of the six ways to assign values to these coefficients, four satisfy this inequality: $1^2 - 4 \cdot 2 \cdot 3$, $1^2 - 4 \cdot 3 \cdot 2$, $2^2 - 4 \cdot 1 \cdot 3$, and $2^2 - 4 \cdot 3 \cdot 1$. Hence the probability is $\frac{4}{6} = \frac{2}{3}$.

6. **−30/11** We have ${}_{5}\overset{2}{\underset{4}{\triangle}}{}_{3} = \dfrac{2 \cdot 3}{4 - 5} = -6$.

Then we compute ${}_{16}\overset{6}{\underset{-6}{\triangle}}{}_{10} = \dfrac{6 \cdot 10}{-6 - 16} = -\dfrac{60}{22} = -\dfrac{30}{11}$.

7. $\boldsymbol{y = 0, x = 1, x = -2}$ The denominator factors as $(x+2)(x-1)$ so the vertical asymptotes are $x = -2$ and $x = 1$. For large values of x, the function approaches $y = 0$, so this is the horizontal asymptote.

8. $\mathbf{-1}$ Call the two numbers r and s. Then $r + s = rs = 1$. Squaring, we get $r^2 + 2rs + s^2 = 1$. It follows that $r^2 + s^2 = 1 - 2rs = 1 - 2 \cdot 1 = -1$.

9. $\mathbf{-1}$ By Viéte's Formulas, we have $m + n = -m$ and $mn = n$. Hence $m = 1$ so the sum is $-m = -1$.

10. **2** We have

$$\begin{aligned}\frac{\sin 210^\circ}{\csc 210^\circ} + \frac{\cos 210^\circ}{\sec 210^\circ} + \tan 210^\circ \cot 210^\circ &= \sin^2 210^\circ + \cos^2 210^\circ + 1\\ &= 1 + 1 = 2.\end{aligned}$$

4 Solutions to the 1983 State Tournament

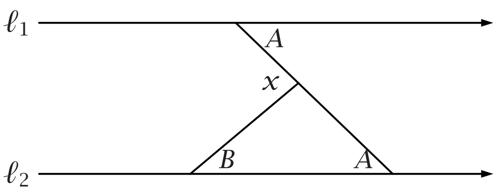
The Written Test Solutions

1. **B** We require the expression in the radical to be positive and defined. Hence, we solve

$$\frac{2x+2}{x-3} > 0.$$

The critical numbers of the expression are $x = -1$ and $x = 3$. These numbers split the real number line into three intervals. Checking the sign of each interval, we find that the domain is all reals x such that $x \leq -1$ or $x > 3$.

2. **B** By alternate interior angles, we have another angle congruent to the angle marked A; this congruent angle is also marled A in the figure below.



The angle marked x is the exterior to the triangle containing the angles marked A and B. Then by the Exterior Angle Theorem, the angle x has measure $x = A + B = 40^\circ + 44^\circ = 84^\circ$.

3. **A** Let the expression be equal to x. Then square both sides to get $12 + x = x^2$. This can be written $x^2 - x - 12 = 0$ which has solutions $x = -3$ and $x = 4$. We want the positive solution, so $x = 4$.

4. **D** Note that we have the following algebraic identity, which we proceed to manipulate.

$$\begin{aligned}
\left(\frac{1}{r^2} + \frac{1}{s^2}\right)^2 &= \frac{1}{r^2} + \frac{2}{rs} + \frac{1}{s^2} \\
\left(\frac{r+s}{rs}\right)^2 &= \frac{1}{r^2} + \frac{1}{s^2} + \frac{2}{rs} \\
\frac{(r+s)^2}{(rs)^2} - \frac{2}{rs} &= \frac{1}{r^2} + \frac{1}{s^2} \\
\frac{(r+s)^2 - 2rs}{(rs)^2} &= \frac{1}{r^2} + \frac{1}{s^2}.
\end{aligned}$$

From Viéte's Formulas, we have $r + s = -\frac{b}{a}$ and $rs = \frac{c}{a}$. Then,

$$\begin{aligned}
\frac{1}{r^2} + \frac{1}{s^2} &= \frac{(r+s)^2 - 2rs}{(rs)^2} = \frac{(-\frac{b}{a})^2 - 2 \cdot \frac{c}{a}}{(\frac{c}{a})^2} \\
&= \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{c^2}{a^2}} = \frac{b^2 - 2ac}{c^2}.
\end{aligned}$$

5. **D** Note that when $x = 0$, we have $y = \pm 2$ and when $y = 0$, we have $x = \pm 4$. The graph is a quadrilateral with vertices $(\pm 4, 0)$ and $(0, \pm 2)$. The region in the first quadrant is therefore a right triangle of legs 2 and 4; the area of this triangle is $\frac{1}{2} \cdot 2 \cdot 4 = 4$. The area of the quadrilateral is therefore $4 \cdot 4 = 16$.

6. **A** Let the polynomial be $p(x)$. By the Remainder Theorem, we have $p(-1) = r$ and $p(-2) = r$ for some real r. Hence, we solve the system of equations

$$\begin{cases} -1 + k - 2 + 7 = r \\ -8 + 4k - 4 + 7 = r, \end{cases} \quad \text{which becomes} \quad \begin{cases} k - r = -4 \\ 4k - r = 5. \end{cases}$$

Subtracting the first from the second yields $3k = 9$. It follows that $k = 3$.

7. **B** The only possibility for the two acute angles are that they are 30° and 60°. Hence the triangle is a 30°-60° right triangle with long leg 6. It follows that the short leg is $6/\sqrt{3} = 2\sqrt{3}$ and the hypotenuse is $4\sqrt{3}$.

8. **D** Consider the number of kilograms produced per man-hour. Three men working three hours per day for three days is a total of $3 \cdot 3 \cdot 3 = 27$ man-hours. Thus, the kilograms are produced at a rate of $\frac{3}{27} = \frac{1}{9}$ kg/mhr. Four men working four hours per day for four hours work a total of $4 \cdot 4 \cdot 4 = 64$ man-hours. Hence, they will produce $64 \cdot \frac{1}{9} = \frac{64}{9}$ kg.

9. **E** Let the first term be a and the common difference be d. The sum of the first six terms is $a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) + (a + 5d) = 6a + 15d$. We are told that $d = 2a$, and $6a + 15d = a^2$. Thus $a^2 = 6a + 15(2a)$ so that $a^2 = 36a$. The only positive solution to this equation is 36.

10. **A** The slope of a line is the tangent of the angle it makes with the x-axis. The given line has slope $\frac{1}{3}$, so this is the tangent of the angle α it makes with the x-axis. The line we want must have an angle of $\alpha + 45°$, so the line's slope must be $\tan(\alpha + 45°)$. It follows that the slope we seek must be

$$\tan(\alpha + 45°) = \frac{\tan\alpha + \tan 45°}{1 - \tan\alpha \tan 45°} = \frac{\frac{1}{3} + 1}{1 - \frac{1}{3} \cdot 1}$$
$$= \frac{\frac{4}{3}}{\frac{2}{3}} = \frac{4}{2} = 2.$$

The only line with slope 2 of our answer choices is $2x - y + 1 = 0$.

11. **D** We require the graph to pass through the point $(1, 2)$. This limits our choices to either C or D. As the graph must be parabolic for all $x > 1$, the correct graph must that in answer choice D.

12. **C** There are $4! = 24$ ways to arrange the four digits. However, a four-digit number cannot begin with the digit 0. There are $3! = 6$ ways to arrange the four digits with leading digit 0. Hence, there are $24 - 6 = 18$ ways to create a four-digit number from the digits 2, 3, 0, and 4.

13. **A** Any set of four consecutive integers will have an integer divisible by 4, an integer divisible by 3, and an integer divisible by 2 but not by 4. Hence, the largest factor of the product will always be $2 \cdot 3 \cdot 4 = 24$.

14. **B** We complete the square on the equation of the circle, we get

$$(x + 2)^2 + (y - 1)^2 = 5$$

and we find the intersection of the lines to be $(-1, 2)$. The distance between the center $(-2, 1)$ of the circle and the point $(-1, 2)$ is $\sqrt{2}$. Thus the center of the ball is $\sqrt{5} - \sqrt{2}$ units from the edge of the hoop. Since $\sqrt{5} - \sqrt{2} < 1$, the ball touches the rim as it goes through the hoop.

15. **C** This is the definition of the derivative of the function $f(x) = \sqrt{x}$ at the point $x = 4$. Since $f'(x) = \frac{1}{2\sqrt{x}}$, we see that the limit is equal to $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$.

16. **D** There are $\binom{5}{2}\binom{3}{1}$ ways of making the selection from the box. As there are $\binom{8}{3}$ ways of selecting any three balls, the probability is

$$\frac{\binom{5}{2}\binom{3}{1}}{\binom{8}{3}} = \frac{10 \cdot 3}{56} = \frac{15}{28}.$$

17. **D** The maximum occurs at the vertex of the parabola $N(t) = 24t(10 - 2t) + 15 = -48t^2 + 240t + 15$. The t-coordinate is $-\frac{b}{2a} = -\frac{240}{-96} = \frac{5}{2}$. Then the maximum is $N(\frac{5}{2}) = 24 \cdot \frac{5}{2}(10 - 5) + 15 = 60 \cdot 5 + 15 = 315$.

18. **D** The maximum occurs at the vertex of the parabola $y(t) = 96t - 16t^2$. The t-coordinate is $-\frac{b}{2a} = -\frac{96}{-32} = 3$. Then the maximum is $y(3) = 96 \cdot 3 - 16 \cdot 9 = 288 - 144 = 144$.

19. **D** Let the two numbers be x and y. Then $x + y = 8$ and we wish to minimize $x^2 + y^2 = x^2 + (8 - x)^2 = 2x^2 - 16x + 64$. The minimum occurs at the vertex of the parabola. The x-coordinate of the vertex is $= \frac{b}{2a} = -\frac{-16}{4} = 4$. Hence, $y = 8 - 4 - 4$ and the numbers are 4 and 4.

20. **B** The increase is the dfference between m and n as a fraction of n. The percentage is this fraction multiplied by 100. Hence the percentage increase is $100(m - n)/n$.

21. **B** We require the magnitude of the vector to be 1. Hence, the

square of the magnitude should be 1, and we solve

$$\begin{aligned}\left(\frac{k-2}{13}\right)^2 + \left(\frac{k+5}{13}\right)^2 &= 1\\ \frac{(k-2)^2+(k+5)^2}{169} &= 1\\ k^2-4k+4+k^2+10k+25 &= 169\\ 2k^2+6k-140 &= 0\\ k^2+3k-70 &= 0\\ (k+10)(k-7) &= 0.\end{aligned}$$

The solutions are $k = -10$ and $k = 7$.

22. **B** The number of diagonals in an n-gon is given by $\frac{1}{2}n(n-3)$. For an octagon, we have $\frac{1}{2} \cdot 8(8-3) = 4 \cdot 5 = 20$.

23. **D** Note that we may rewrite the given functional equation to $F(n+1) - F(n) = \frac{1}{4}$. Since the difference between consecutive values is constant, the function F must be linear with slope $\frac{1}{4}$. Hence, the function $F(n)$ may be represented by $F(n) = \frac{1}{4}(n - 101) + 27$. It follows that $F(1) = \frac{1}{4}(-100) + 27 = 2$.

24. **D** We find a parametrization of the segement from P_1 to P_2. Since $P_2 - P_1 = (6,-3) - (1,7) = (5,-10)$, we have the paramatrization $x = 5t + 1$ and $y = -10t + 7$ for $0 \le t \le 1$. To find the point which splits the segment in the ratio 2 : 3, we use $t = \frac{2}{5}$ in the parametric equations: $(x, y) = (5 \cdot \frac{2}{5} + 1, -10 \cdot \frac{2}{5} + 7) = (3, 3)$.

25. **E** The angle whose sine is $-\frac{1}{2}$ and whose cosine is $-\frac{\sqrt{3}}{2}$ would seem to be 210°. However, we were never told that the angle we want is a unit circle angle; that is, we were not told that the angle had to be between 0° and 360°. So we cannot definitively say what the angle $2x$, and therefore the angle x, is.

26. **C** We trace through the program in the following table.

N	A	B
	2	1
1	2	1
2	2	2
3	4	2
4	8	4
5	32	8

It follows that the number printed is 8.

27. **C** The distance k from a point (x, y) to the line is given by

$$\frac{|12x - 5y - 15|}{\sqrt{12^2 + 5^2}} = k$$
$$\frac{|12x - 5y - 15|}{13} = k$$
$$12x - 5y - 15 = \pm 13k.$$

Thus an equation of the line parallel to the given line is $12x - 5y + 13k - 15 = 0$.

28. **E** Note that $y = x$ is a solution to the equation; this implies that $x - y$ is a factor. Factoring yields $(x - y)(x + 2y - 1) = 0$, so this equation represents two intersecting lines.

29. **D** The slope of the line is the tangent of the angle the line makes with the x-axis. Let α be the angle L_2 makes with the x-axis, so that its slope is $\tan\alpha$. Let β be the angle L_1 makes with the x-axis, so that its slope is $\tan\beta = \frac{2}{3}$. Then $\alpha - \beta = 45°$, and so

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$
$$\tan(45°) = \frac{\tan\alpha - \frac{2}{3}}{1 + \frac{2}{3}\tan\alpha}$$
$$1 = \frac{3\tan\alpha - 2}{3 + 2\tan\alpha}$$
$$3 + 2\tan\alpha = 3\tan\alpha - 2$$
$$5 = \tan\alpha.$$

Hence, the slope of line L_2 is 5. (*Note:* The preceeding work implies that the line L_2 has the larger angle with the x-axis than L_1. However, it could be that L_2 has a smaller angle than L_1; indeed, this would imply we want $\tan(\beta - \alpha)$. This leads to a slope of $-\frac{1}{5}$, and this is not an answer choice. So we are forced to conclude that the angle L_2 makes is larger than that of L_1.)

30. **C** The equations are those of a circle centered at the origin and an upward-facing parabola. There will be two intersections.

31. **D** Using the property $\log\frac{a}{b} = \log a - \log b$, we see that x becomes $\log 2 - \log 1 + \log 3 - \log 2 + \log 4 - \log 3 + \cdots + \log 100 - \log 99 = \log 100 - \log 1 = 2 - 0 = 2$.

32. **D** The space diagonal is given by $11 = \sqrt{2^2 + 6^2 + h^2} = \sqrt{40 + h^2}$. Hence, $h^2 = 11^2 - 40 = 81$ so that $h = 9$. The volume is $2 \cdot 6 \cdot 9 = 108$.

33. **E** Note that the angle adjacent to the 100°-angle is congruent to θ. Thus $\theta = 180° - 100° = 80°$.

34. **B** We have

$$\sqrt[3]{x\sqrt[4]{x\sqrt[3]{x}}} = \left(x\left(x\left(x^{1/3}\right)\right)^{1/4}\right)^{1/3} = \left(x\left(x^{4/3}\right)^{1/4}\right)^{1/3}$$
$$= \left(x\left(x^{1/3}\right)\right)^{1/3} = \left(x^{4/3}\right)^{1/3} = x^{4/9} = \sqrt[9]{x^4}.$$

35. **D** The only number not in the range of $g(x)$ is the value of the horizontal asymptote. As x increases without bound, $g(x)$ approaches $\frac{2}{3}$. This is the number not in the range.

36. **B** Completing the square on both equations yields

$$(x-2)^2 + (y-3)^2 = 25$$

and

$$(x-1)^2 + (y-2)^2 = 1.$$

The distance between the centers of the circles is $\sqrt{2}$ and the smallest distance from the center of the smaller circle to the edge of the larger circle is $5 - \sqrt{2} > 1$. Thus the smaller circle is completely contained in the larger circle. It follows that the area sought is $25\pi - \pi = 24\pi$.

37. **B** Note that $\sin\alpha = \frac{1}{3}$ implies that $\cos\alpha = \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}$. Also, $\tan\beta = 1$ implies that $\sin\beta = \cos\beta = \frac{\sqrt{2}}{2}$. Using the cosine angle sum identity, we have

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{2\sqrt{2}}{3}\cdot\frac{\sqrt{2}}{2} - \frac{1}{3}\cdot\frac{\sqrt{2}}{2}$$
$$= \frac{4}{6} - \frac{\sqrt{2}}{6} = \frac{4-\sqrt{2}}{6}.$$

38. **B** Multiplying both sides of the given indentity by $x(x-1)(x+1)$ yields

$$5x + 1 = A(x-1)(x+1) + Bx(x-1) + Cx(x+1).$$

Since the identity must be true for all values of x, we let $x = 0$ to get $A = -1$. Let $x = -1$ to get $-4 = B(-1)(-2)$, which gives $B = -2$. Finally, let $x = 1$ to get $6 = 2C$, or $C = 3$. Hence, $A + B + C = -1 - 2 + 3 = 0$.

39. **A** Solving for r^2 we get $r^2 = \frac{V}{\pi h}$. Taking the logarithm of both sides yields

$$\log(r^2) = \log\left(\frac{V}{\pi h}\right)$$
$$2\log r = \log V - (\log \pi + \log h)$$
$$\log r = \frac{\log V - \log \pi - \log h}{2}.$$

40. **E** The slope of a line is equal to the tangent of the angle it makes with the x-axis. Since $\tan 120^\circ = -\sqrt{3}$, the slope of the line we seek is $-\sqrt{3}$. Since we are told this line ankes the angle 120° with the positive x-axis, the line makes a 30°-60° right triangle with the axes. The distance from the origin (the right angle of our triangle) to the line must be 5. Thus the length along the positive x-axis to the point of intersection with the line must be $2 \cdot 5/\sqrt{3} = 10/\sqrt{3}$. Therefore the equation of the line is $y = -\sqrt{3}(x - \frac{10}{\sqrt{3}}) = -\sqrt{3}x + 10$, which can be rewritten as $\sqrt{3}x + y - 10 = 0$.

41. **C** We have

$$\log_4 x - \frac{1}{\log_4 x} = \frac{3}{2}$$
$$(\log_4 x)^2 - \frac{3}{2}\log_4 x - 1 = 0$$
$$2(\log_4 x)^2 - 3\log_4 x - 1 = 0$$
$$(2\log_4 x + 1)(\log_4 x - 2) = 0.$$

Hence, either $\log_4 x = -\frac{1}{2}$ or $\log_4 x = 2$. It follows that $x = 4^{-1/2} = \frac{1}{2}$ or $x = 4^2 = 16$.

42. **C** The angles of any pentagon, whether convex or concave, is $3 \cdot 180^\circ = 540^\circ$.

43. **D** Consider a conic section with equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

This conic's axes have been rotated through an angle θ where θ is given by

$$\tan 2\theta = \frac{B}{A - C}.$$

For the conic in our problem, we have

$$\tan 2\theta = \frac{-6\sqrt{3}}{7 - 13} = \frac{-6\sqrt{3}}{-6} = \sqrt{3}.$$

This implies that $2\theta = 60^\circ$, so that $\theta = 30^\circ$.

44. **B** We use the determinant method. We have

$$\frac{1}{2}\begin{vmatrix} -5 & -2 & 2 & 5 & 2 & -5 \\ -2 & 5 & 7 & 1 & -4 & -2 \end{vmatrix} = \frac{1}{2}|-25 - 14 + 2 - 20 - 4 - (4 + 10 + 35 + 2 + 20)| = \frac{1}{2}|-61 - 71| = \frac{1}{2} \cdot 132 = 66.$$

(*Note:* A polygon whose vertices are lattice points must have area which is half an integer, so answer choices A, C, and D are immediately eliminated.)

45. **B** Let a be the first term and let d be the common difference. Then the second term is $a + d = 4$ and the ninth term is $a + 8d = -17$. Subtracting equations gives us $7d = -21$ so that $d = -3$.

46. **B** We have $\dfrac{\frac{1}{2} + \frac{25}{2}}{2} - \sqrt{\dfrac{1}{2} \cdot \dfrac{25}{2}} = \dfrac{13}{2} - \dfrac{5}{2} = \dfrac{8}{2} = 4$.

47. **C** Since tangents and radii are perpendicular, we have a right triangle whose hypotenuse is the segment connecting the center and A and whose legs are a radius and the tangent segment. The lengths of the legs are r and $\frac{4}{3}r$, and the length of the hypotenuse is $r + x$, where x is the length from A to the circle. By the Pythagorean Theorem, we have

$$\begin{aligned} r^2 + \left(\frac{4}{3}r\right)^2 &= (r + x)^2 \\ r^2 + \frac{16}{9}r^2 &= (r + x)^2 \\ \frac{25}{9}r^2 &= (r + x)^2 \\ \frac{5}{3}r &= r + x \\ x &= \frac{2}{3}r = \frac{2}{3} \cdot \frac{3}{4}\ell = \frac{1}{2}\ell. \end{aligned}$$

48. **B** There are seven letters in the word, with three As and two Ts. Hence, the number of different arrangements is

$$\frac{7!}{3!2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 2} = 7 \cdot 5 \cdot 4 \cdot 3 = 20 \cdot 21 = 420.$$

49. **B** There are eight prime numbers between 1 and 20: 2, 3, 5, 7, 11, 13, 17, and 19. Out of 20 numbers, the probability is $\frac{8}{20} = \frac{2}{5}$.

50. **E** The radius of the circle is 2 because the area is given as 4π. Since the triangle is equilateral, the non-circular sides of the shaded region are each the same length as the radius. The circular edge of the shaded region is exactly one-sixth of the circumference of the circle. Hence, the perimeter is $\frac{1}{6}\cdot 4\pi + 4\cdot 2 = \frac{2\pi}{3} + 8$.

The Ciphering Solutions

1. **17** There are twelve squares of unit area and five squares of area 4 units. Hence there are 17 squares.

2. **−1, 0, 1** Clearly, a cannot be 1 or -1. Moreover, if $a = 0$, then the entire denominator is 0, which cannot happen. Thus a cannot be -1, 0, or 1.

3. **39π/2** The diameter of the large circle is $3 + 8 + 3 = 14$, so its radius is 7, and thus its area is 49π. The area of the circle of diameter 8 is 16π, and the six circles of diameter 3 each have area $\frac{9}{4}\pi$. The shaded region's area is therefore $49\pi - 16\pi - 6 \cdot \frac{9}{4}\pi = 33\pi - \frac{27}{2}\pi = \frac{39}{2}\pi$.

4. **16/65** Let $\alpha = \text{Arcsin}\,\frac{4}{5}$ and $\beta = \text{Arctan}\,\frac{5}{12}$ so that $\sin\alpha = \frac{4}{5}$ and $\tan\beta = \frac{5}{12}$. Then $\sin\alpha = \frac{4}{5}$ implies that $\cos\alpha = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$. Also, $\tan\beta = \frac{5}{12}$ implies that $\sin\beta = \frac{5}{13}$ and $\cos\beta = \frac{12}{13}$. Using the cosine angle sum identity, we have

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{3}{5}\cdot\frac{12}{13} - \frac{4}{5}\cdot\frac{5}{13}$$
$$= \frac{36 - 20}{65} = \frac{16}{65}.$$

5. **1/9** The odds that Batman beats Superman are 2 : 1. The odds that Wonder Woman beats Batman are 3 : 1 = 6 : 2. Hence, the odds of all three are 6 : 2 : 1. Thus the probability Superman wins is $\frac{1}{6+2+1} = \frac{1}{9}$.

6. **1** The only such function with this property is a linear function of the form $f(x) = ax$ for real constant a. Since $f(8) = 2$, we have $2 = 8a$ so that $a = \frac{1}{4}$. Then $f(4) = \frac{1}{4} \cdot 4 = 1$.

7. $(\boldsymbol{a}+\boldsymbol{b})/(\boldsymbol{a}+\mathbf{8}\boldsymbol{b})$ We have

$$\frac{1-\dfrac{1}{\dfrac{a}{b}+2}}{1+\dfrac{3}{\dfrac{a}{2b}+1}}=\frac{1-\dfrac{b}{a+2b}}{1+\dfrac{6b}{a+2b}}=\frac{1+2b-b}{a+2b+6b}=\frac{a+b}{a+8b}.$$

8. **72** Call the intersection of the diagonals F. Note that $\triangle AFB$ and $\triangle ABC$ are isosceles. Since $m\angle B = 108^\circ$, then $m\angle CAB = m\angle ACB = 36^\circ$. Hence $m\angle AFB = 108^\circ$, and so $m\angle 2 = m\angle CFB = 180^\circ - 108^\circ = 72^\circ$.

9. **1936** Since $43^2 = 1849$, the next perfect square year is $44^2 = 1936$.

10. **64** We have

$$\begin{aligned}\log_2(\log_3(\log_4 x)) &= 0\\ \log_3(\log_4 x) &= 2^0 = 1\\ \log_4 x &= 3^1 = 3\\ x &= 4^3 = 64.\end{aligned}$$

Part III

The Answers

5

The Answers to the State Tournaments

On the following pages are the *answers* to all the problems. This section is here for those who wish to simply check their work before reading the solutions. This also makes a handy answer key for teachers using these problems for review or enrichment.

Answers to the 1982 State Tournament

Written Test

1. D	11. B	21. B	31. B	41. C
2. B	12. E	22. B	32. B	42. C
3. D	13. C	23. A	33. E	43. D
4. A	14. A	24. A	34. D	44. D
5. D	15. C	25. B	35. C	45. B
6. B	16. E	26. A	36. C	46. B
7. A	17. B	27. C	37. B	47. B
8. D	18. C	28. C	38. E	48. D
9. D	19. A	29. D	39. E	49. A
10. D	20. A	30. C	40. C	50. B

Ciphering

1. 60
2. $\frac{\sqrt{13}}{13}$
3. $(x-3)^2 + (y-3)^2 = 18$
4. 0
5. $\frac{2}{3}$
6. $-\frac{30}{11}$
7. $y = 0, x = 1, x = -2$
8. -1
9. -1
10. 2

Answers to the 1983 State Tournament

Written Test

1. B	11. D	21. B	31. D	41. C
2. B	12. C	22. B	32. D	42. C
3. A	13. A	23. D	33. E	43. D
4. D	14. B	24. D	34. B	44. B
5. D	15. C	25. E	35. D	45. B
6. A	16. D	26. C	36. B	46. B
7. B	17. D	27. C	37. B	47. C
8. D	18. D	28. E	38. B	48. B
9. E	19. D	29. D	39. A	49. B
10. A	20. B	30. C	40. E	50. E

Ciphering

1. 17
2. $-1, 0, 1$
3. $\frac{39}{2}\pi$
4. $\frac{16}{65}$
5. $\frac{1}{9}$
6. 1
7. $\frac{a+b}{a+8b}$
8. 72
9. 1936
10. 64

Index

In addition to being a traditional index of key words, this index also serves as a collection of problems by topic. Each problem in this book has been placed under a subject-specific topic (or two, or three), such as “logarithms” or “probability, geometric”. It is by these topics that the index is arranged. This should make it easier for teachers to target review or enrichment for students in certain areas. The entry format of the topics is “round, year, problem number, *page number*”. Thus, one can find that there is a test problem involving angles on the 1983 written test: problem number 2 on page 13.